Continued Fraction of $\sqrt[N]{M}$

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Abstract

Let $\sqrt[N]{M}$ be an algebraic number of order $N \geq 3$, and let $r_0, r_1, \ldots$ be the sequence of remainders of its continued fraction. Hypothesis: The sequence

$$\xi_n = \log_2(1 + r_n)$$

has the uniform distribution.

Introduction

In the paper we present a reliable hypothesis about the remainders of the continued fraction of the algebraic number $\sqrt[N]{M}$ of order $N \geq 3$. From the hypothesis it follows that the continued fraction has an unbounded sequence of elements. In particular, this gives the answer to the Kchinchin problem [1] about the continued fractions of algebraic numbers.

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1. Continued fractions

Let $x > 0$. Then

$$x = a_0 + r_0 = a_0 + \cfrac{1}{a_1 + r_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + r_2}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + r_3}}} = \cdots$$

where

$$\begin{aligned} a_0 &= [x] \\ r_0 &= x - [x] \end{aligned}$$

and

$$\begin{aligned} a_n &= [1/r_{n-1}]\,, \\ r_n &= 1/r_{n-1} - [1/r_{n-1}]\,. \end{aligned}$$

for $n = 1, 2 \ldots$. The real numbers $r_0, r_1, \ldots$ are called the remainders, and the integer numbers $a_0, a_1, \ldots$ are called the elements of the continued fraction.

Continued fractions of rational numbers are finite.

$$\frac{3}{5} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued fractions of algebraic numbers of order 2 are periodic.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

Continued fractions of algebraic numbers of order ≥ 3 are chaotic.

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{1 + \cfrac{1}{14 + \cdots}}}}}}}}}}}$$

The first 100 000 elements of the last continued fraction look like a stochastic process between 1 and 532 220.

2. Gauss theorem

Let x be a random variable with the uniform distribution on $[0,1]$. For each $n=0,1,2,\dots$ the remainder r_n is a random variable on $[0,1]$. Let $f_n(t)$ be the probability density of the random variable r_n. Since $r_0=x$, we have $f_0(t)=1$. Then

$$f_n(t)=\sum_{k=1}^{\infty}\frac{1}{(t+k)^2}f_{n-1}\left(\frac{1}{t+k}\right).$$

From

$$\frac{1}{1+t}=\sum_{k=1}^{\infty}\frac{1}{(t+k)^2}\frac{1}{1+\dfrac{1}{t+k}}$$

and

$$\int_0^1 f_n(t)\,dt=1$$

it may be proved that

$$\lim_{n\to\infty}f_n(t)=\frac{1}{\ln 2}\frac{1}{1+t}.$$

The proof of the Gauss theorem is in [1].

Thus the random variables

$$\xi_n=\log_2(1+r_n)$$

converge to the uniform distribution. This leads to a reliable hypothesis about the remainders of the continued fraction of the algebraic number $\sqrt[N]{M}$.

3. Remainders of the continued fraction

We need the notion of the density of a sequence [2]. Let $x_1,x_2,\dots$ be a sequence on $[0,1]$. Suppose that for each $0\le t\le 1$ there exists the limit

$$F(t)=\lim_{n\to\infty}\frac{m_n(t)}{n},$$

where $m_n(t)$ is the number of points $x_1,\dots,x_n$ such that $0\le x_k\le t$. The function

$$f(t)=\frac{d}{dt}F(t)$$

is called the density of the sequence $x_1,x_2,\dots$. In the case $f(t)=1$ we say that the sequence has the uniform distribution.

For example, for any irrational x the sequence

$$x_n=nx-[nx]$$

has the uniform distribution.

Suppose that the irrational algebraic number $\sqrt[N]{M}$ is expanded in the continued fraction

$$\sqrt[N]{M} = a_0 + r_0 = a_0 + \cfrac{1}{a_1 + r_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + r_2}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + r_3}}} = \cdots$$

Hypothesis. If $N \geq 3$, then the sequence

$$\xi_n = \log_2(1 + r_n)$$

has the uniform distribution.

Example. Let us check the hypothesis for the number $\sqrt[3]{2}$. By direct calculation, we obtain the first 1000 elements $\xi_0, \xi_1, \ldots, \xi_{999}$ and see the expected identities

$$\frac{1}{1000} \sum_{k=0}^{999} \xi_k = 0.501912... \approx \frac{1}{2}$$

and

$$\frac{1}{1000} \sum_{k=0}^{999} \xi_k^2 = 0.332463... \approx \frac{1}{3}.$$

In the table

1	2	3	4	5	6	7	8	9	10
90	98	109	102	101	89	115	96	112	88

we see how many elements $\xi_0, \xi_1, \ldots, \xi_{999}$ are in the intervals

$$\left[\frac{k-1}{10}, \frac{k}{10}\right]$$

where $k = 1, 2, \ldots, 10$.

4. Elements of the continued fraction

The chaotic behavior of the remainders $r_0, r_1, \ldots$ leads to the chaotic behavior of the elements $a_0, a_1, \ldots$. Nevertheless the elements may be calculated by means of recurrence relations.

The numerator and denominator of the kth convergent are calculated as follows

$$\begin{aligned} P_k &= a_k P_{k-1} + P_{k-2}, \\ Q_k &= a_k Q_{k-1} + Q_{k-2}, \end{aligned}$$

where

$$\begin{array}{rcl} P_{-1} = 1 & \text{and} & P_0 = a_0 \\ Q_{-1} = 0 & & Q_0 = 1 \end{array}$$

It seems to be true that for any $\sqrt[N]{M}$ there exists n such that

$$a_k = \left[-\frac{(P_{k-2}^N - MQ_{k-2}^N)P_{k-1}^{N-1}}{(P_{k-1}^N - MQ_{k-1}^N)P_{k-2}^{N-1}} \right].$$

for all $k \geq n+1$. For $3 \leq N \leq 5$ and $2 \leq M \leq 10$ such n may be taken from the table.

	2	3	4	5	6	7	8	9	10
3	1	2	3	3	3	3	$\emptyset$	1	1
4	2	2	2	3	4	3	4	2	5
5	1	2	2	2	3	2	3	3	5

Example. For $\sqrt[3]{2}$ we have $n = 1$. We put

$$\begin{array}{rcl} a_0 = 1 & & a_1 = 3 \\ P_0 = 1 & \text{and} & P_1 = 4 \\ Q_0 = 1 & & Q_1 = 3 \end{array}$$

Then

$$\begin{aligned} a_k &= \left[-\frac{(P_{k-2}^3 - 2Q_{k-2}^3)P_{k-1}^2}{(P_{k-1}^3 - 2Q_{k-1}^3)P_{k-2}^2} \right], \\ P_k &= a_k P_{k-1} + P_{k-2}, \\ Q_k &= a_k Q_{k-1} + Q_{k-2} \end{aligned}$$

for all $k = 2, 3 \ldots$. The recurrence relations give the elements

1 3 1 5 1 1 4 1 1 8 1 14 1 10 2 1 4 12 2 3 2 1 3 4 1 1 2 14 3 12 1 ...

of the continued fraction of the algebraic number $\sqrt[3]{2}$.

Bibliography

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[2] G.Polya, G.Szegö. Aufgaben und Lehrsätze aus der Analysis. Sprnger-Verlag, Berlin, 1964

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