Estimations of derivatives of n-convex functions

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Abstract

Estimations of the kind

$$\begin{aligned}
\|u' - u_0'\|_{C[a+\delta,b-\delta]} &\leq A\|u-u_0\|_{C[a,b]}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1},\\
\|u^{(k)} - u_0^{(k)}\|_{C[a+\delta,b-\delta]} &\leq C\|u-u_0\|_{C[a,b]}^{(n-k)/n},\\
\|u-u_0\|_{C[a+\delta,b-\delta]} &\leq C\|u-u_0\|_{L_1[a,b]}^{n/(n+1)}
\end{aligned}$$

are obtained for n-convex functions.

Introduction

1. Estimation of the first derivative
2. Estimation of higher derivatives
3. Estimation of C-norm by L-norm

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Introduction

A function $u(x)$ is called n-convex on the interval $[a,b]$ if

$$\begin{vmatrix}
1 & 1 & \cdots & 1\\
x_0 & x_1 & \cdots & x_n\\
 & & \cdots & \\
x_0^{n-1} & x_1^{n-1} & \cdots & x_n^{n-1}\\
u(x_0) & u(x_1) & \cdots & u(x_n)
\end{vmatrix} \geq 0$$

for all $a < x_0 < x_1 < \ldots < x_n < b$.

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An 1-convex function is increasing and a 2-convex function is convex. If $n \geq 2$, then an n-convex function $u(x)$ is C^{n-2}-differentiable. By the Popoviciu theorem, a C^{n-2}-differentiable $u(x)$ is n-convex if and only if $u^{(n-2)}(x)$ is convex [1]. In particular, a C^n-differentiable $u(x)$ is n-convex if and only if $u^{(n)}(x)$ is positive. The article is devoted to the estimation of derivatives of C^n-differentiable n-convex functions. Analogous estimations are given in the paper [2].

1. Estimation of the first derivative

Theorem 1 *Let $n \geq 2$. Let $u_0(x)$ be n-convex and C^n-differentiable on the interval $[a,b]$. There are exist A and B such that any n-convex $u(x)$ satisfies the inequality*

$$\|u' - u_0'\|_{C[a+\delta,b-\delta]} \leq A\|u - u_0\|_{C[a,b]}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1},$$

where $0 < \delta < (b-a)/n$.

Proof. Consider the case $n = 2$. Let us take a point

$$a + \delta \leq x_0 \leq b - \delta$$

and define the functions

$$\begin{aligned} v_0(x) &= u_0(x) - u_0(x_0) - u_0'(x_0)(x - x_0), \\ v(x) &= u(x) \ \ - u_0(x_0) - u_0'(x_0)(x - x_0). \end{aligned}$$

We see that

$$v_0(x_0) = v_0'(x_0) = 0.$$

Since $v(x)$ is convex, we have

$$v(x_0) - v(x_0 - \delta) \leq v'(x_0 - 0)\delta \leq v'(x_0 + 0)\delta \leq v(x_0 + \delta) - v(x_0).$$

Let

$$\varepsilon = \|v - v_0\|_{C[a,b]} = \|u - u_0\|_{C[a,b]}.$$

Then

$$v(x_0) - v(x_0 - \delta) \geq -2\varepsilon + v_0(x_0) - v_0(x_0 - \delta) = -2\varepsilon - \frac{1}{2}v_0''(x_0 - \Theta_-)\delta^2$$

and

$$v(x_0 + \delta) - v(x_0) \leq +2\varepsilon + v_0(x_0 + \delta) - v_0(x_0) = +2\varepsilon + \frac{1}{2}v_0''(x_0 + \Theta_+)\delta^2,$$

where $0 < \Theta_-, \Theta_+ < \delta$. We see that

$$\begin{aligned} u'(x_0 - 0) - u_0'(x_0) &= v'(x_0 - 0), \\ u'(x_0 + 0) - u_0'(x_0) &= v'(x_0 + 0). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
u'(x_0-0)-u_0'(x_0) &\geq -2\varepsilon\delta^{-1}-\frac{1}{2}\|u_0''\|_{C[a,b]}\delta,\\
u'(x_0+0)-u_0'(x_0) &\leq +2\varepsilon\delta^{-1}+\frac{1}{2}\|u_0''\|_{C[a,b]}\delta.
\end{aligned}$$

Thus

$$|u'(x_0)-u_0'(x_0)| \leq 2\varepsilon\delta^{-1}+\frac{1}{2}\|u_0''\|_{C[a,b]}\delta.$$

This means that

$$\|u'-u_0'\|_{C[a+\delta,b-\delta]} \leq A\|u-u_0\|_{C[a,b]}\delta^{-1}+B\|u_0''\|_{C[a,b]}\delta$$

with $A=2$ and $B=1/2$.

Consider the case $n\geq 3$. Let us take a point

$$a+\delta \leq x_0 \leq b-\delta.$$

Consider integers $p,q\geq 1$ satisfying the inequality

$$a+p\delta \leq x_0 \leq b-q\delta.$$

Since $(p+q)\delta \leq b-a$ and $n\delta \leq b-a$, we may assume that $p+q=n$.

Consider n-convex functions

$$v_0(x)=u_0(x)-\sum_{k=0}^{n-1}\frac{1}{k!}u_0^{(k)}(x_0)(x-x_0)^k$$

and

$$v(x)=u(x)-\sum_{k=0}^{n-1}\frac{1}{k!}u_0^{(k)}(x_0)(x-x_0)^k.$$

We note that

$$v_0(x_0)=v_0'(x_0)=\ldots=v_0^{(n-1)}(x_0)=0.$$

Suppose that $p+n+1$ is even. Let

$$t_k=\begin{cases} x_0-(p+k)\delta & \text{for} \quad 0\leq k\leq p,\\ x_0+h & \text{for} \quad k=p+1,\\ x_0+k\delta & \text{for} \quad p+2\leq k\leq n,\end{cases}$$

where $0<h<\delta$. Since $v(x)$ is n-convex, we have

$$\begin{vmatrix}
1 & \cdots & 1 & 1 & \cdots & 1\\
t_0 & \cdots & x_0 & x_0+h & \cdots & t_n\\
t_0^2 & \cdots & x_0^2 & (x_0+h)^2 & \cdots & t_n^2\\
 & \cdots & & & \cdots & \\
t_0^{n-1} & \cdots & x_0^{n-1} & (x_0+h)^{n-1} & \cdots & t_n^{n-1}\\
v(t_0) & \cdots & v(x_0) & v(x_0+h) & \cdots & v(t_n)
\end{vmatrix}
=
\begin{vmatrix}
\cdots & 1 & 1 & \cdots\\
\cdots & x_0 & x_0+h & \cdots\\
\cdots & x_0^2 & x_0^2+2hx_0+h^2 & \cdots\\
\cdots & & & \cdots\\
\cdots & x_0^{n-1} & x_0+(n-1)hx_0^{n-2}+\cdots & \cdots\\
\cdots & v(x_0) & v(x_0)+hv'(x_0+\Theta_h) & \cdots
\end{vmatrix}
\geq 0,$$

where $0 < \Theta_h < h$. We put $h \searrow 0$ and obtain

$$\begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ t_0 & \cdots & t_p & 1 & t_{p+2} & \cdots & t_n \\ t_0^2 & \cdots & t_p^2 & 2x_0 & t_{p+2}^2 & \cdots & t_n^2 \\ & \cdots & & & & \cdots & \\ t_0^{n-1} & \cdots & t_p^{n-1} & (n-2)x_0^{n-2} & t_{p+2}^{n-1} & \cdots & t_n^{n-1} \\ v(t_0) & \cdots & v(t_p) & v'(x_0) & v(t_{p+2}) & \cdots & v(t_n) \end{vmatrix} \geq 0.$$

Without loss of generality it can be assumed that $x_0 = 0$. Hence

$$\begin{vmatrix} 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ -p\delta & \cdots & -\delta & 1 & \delta & \cdots & (q-1)\delta \\ (-p)^2 & \cdots & (-1)^2 & 0 & 1^2 & \cdots & (q-1)^2 \\ & \cdots & & & & \cdots & \\ (-p)^{n-1} & \cdots & (-1)^{n-1} & 0 & 1^{n-1} & \cdots & (q-1)^{n-1} \\ v(-p\delta) & \cdots & v(-\delta) & v'(0) & v(\delta) & \cdots & v((q-1)\delta) \end{vmatrix} \geq 0.$$

Since $p+n+1$ is even, we have

$$\delta v'(0) \geq \sum_{k=-p}^{q-1} A_k v(k\delta)$$

with some constant $A_{-p},\ \ldots,\ A_{-1},\ A_0 = 0,\ A_1,\ \ldots,\ A_{q-1}$. We note that

$$v(x) = u(x) - u_0(x) + v_0(x)$$

and

$$v_0(k\delta) = \frac{k^n}{n!} v_0^{(n)}(\Theta_k)\delta^n = \frac{k^n}{n!} u_0^{(n)}(\Theta_k)\delta^n,$$

where $-p\delta < \Theta_k < (q-1)\delta$. Therefore,

$$\delta v'(0) \geq \sum_{k=-p}^{q-1} A_k(u(k\delta) - u_0(k\delta)) + \sum_{k=-p}^{q-1} A_k \frac{k^n}{n!} u_0^{(n)}(\Theta_k)\delta^n.$$

This proves the inequality

$$\delta v'(x_0) \geq -\left(A\|u - u_0\|_{C[a,b]} + B\|u_0^{(n)}\|_{C[a,b]}\delta^n \right).$$

We put

$$t_k = \begin{cases} x_0 - (p-1+k)\delta & \text{for} \quad 0 \leq k \leq p-1, \\ x_0 + h & \text{for} \quad k = p, \\ x_0 + k\delta & \text{for} \quad p+1 \leq k \leq n \end{cases}$$

and analogously obtain the inequality

$$\delta v'(x_0) \leq A\|u - u_0\|_{C[a,b]} + B\|u_0^{(n)}\|_{C[a,b]}\delta^n.$$

This means that

$$\|u' - u_0'\|_{C[a+\delta,b-\delta]} \leq A\|u - u_0\|_{C[a,b]}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1},$$

where $0 < \delta < (b-a)/n$.

The case with odd $p + n + 1$ is considered analogously. The theorem is proved. •

Corollary. There exists A such that any n-convex $u(x)$ satisfies the inequality

$$\|u'\|_{C[a+\delta,b-\delta]} \leq A\|u\|_{C[a,b]},$$

where $0 < \delta < (b-a)/n$.

2. Estimation of higher derivatives

Theorem 2 *Let $n \geq 2$. Let $u_0(x)$ be n-convex and C^n-differentiable on the interval $[a,b]$. For any $R > 0$ and $0 < \Delta < (b-a)/2$ there exists $C > 0$ such that any n-convex $u(x)$ from the ball*

$$\|u - u_0\|_{C[a,b]} \leq R$$

satisfies the inequality

$$\|u^{(k)} - u_0^{(k)}\|_{C[a+\Delta,b-\Delta]} \leq C\|u - u_0\|_{C[a,b]}^{(n-k)/n},$$

where $1 \leq k \leq n-1$.

Proof. Consider the case $k = 1$. Let

$$0 < r < \min\left\{\left(\frac{b-a}{n}\right)^n, \Delta^n\right\}.$$

Assume that

$$\|u - u_0\|_{C[a,b]} \leq r.$$

We put

$$\delta = \|u - u_0\|_{C[a,b]}^{1/n}$$

and see that $\delta < \Delta$ and $\delta < (b-a)/n$. Let

$$C' = A + B\|u_0^{(n)}\|_{C[a,b]},$$

where A and B are taken from Theorem 1.

By Theorem 1, we have

$$\begin{aligned}\|u' - u_0'\|_{C[a+\Delta,b-\Delta]} &\leq \|u' - u_0'\|_{C[a+\delta,b-\delta]} \\ &\leq A\|u-u_0\|_{C[a,b]}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1} \\ &= \left(A + B\|u_0^{(n)}\|_{C[a,b]}\right)\|u-u_0\|_{C[a,b]}^{(n-1)/n} \\ &= C'\|u-u_0\|_{C[a,b]}^{(n-1)/n}.\end{aligned}$$

Hence the inequality

$$\|u-u_0\|_{C[a,b]} \leq r$$

implies the inequality

$$\|u' - u_0'\|_{C[a+\Delta,b-\Delta]} \leq C'\|u-u_0\|_{C[a,b]}^{(n-1)/n}.$$

If $R \leq r$, the statement is proves. Let $R > r$. Assume that

$$r < \|u-u_0\|_{C[a,b]} \leq R.$$

Let

$$\delta = \frac{1}{2}\min\left\{\frac{b-a}{n}, \Delta\right\}$$

and

$$C'' = AR^{1/n}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1}r^{(1-n)/n}.$$

By Theorem 1, we have

$$\begin{aligned}\|u' - u_0'\|_{C[a+\Delta,b-\Delta]} &\leq \|u' - u_0'\|_{C[a+\delta,b-\delta]} \\ &\leq A\|u-u_0\|_{C[a,b]}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1} \\ &= \left\{A\|u-u_0\|_{C[a,b]}^{1/n}\delta^{-1} + B\|u_0^{(n)}\|_{C[a,b]}\delta^{n-1}\|u-u_0\|_{C[a,b]}^{(1-n)/n}\right\}\|u-u_0\|_{C[a,b]}^{(n-1)/n} \\ &\leq C''\|u-u_0\|_{C[a,b]}^{(n-1)/n}.\end{aligned}$$

Let $C = \max\{C', C''\}$. Then the inequality

$$\|u-u_0\|_{C[a,b]} \leq R$$

implies the inequality

$$\|u' - u_0'\|_{C[a+\Delta,b-\Delta]} \leq C\|u-u_0\|_{C[a,b]}^{(n-1)/n},$$

where C depends only on Δ, R and u_0.

Consider the case $2 \leq k \leq n-1$. Since the derivative of an n-convex function is $(n-1)$-convex, we prove by induction.

Let

$$0 = \Delta_0 < \Delta_1 < \Delta_2 < \ldots < \Delta_{n-2} < \Delta_{n-1} = \Delta.$$

We put

$$R_0 = R$$

and define C_0 in such a way that any n-convex function $u(x)$ satisfying the inequality

$$\|u-u_0\|_{C[a+\Delta_0,b-\Delta_0]} \le R_0$$

satisfies the inequality

$$\|u'-u_0'\|_{C[a+\Delta_1,b-\Delta_1]} \le C_0\|u-u_0\|^{(n-1)/n}_{C[a+\Delta_0,b-\Delta_0]}.$$

Then we put

$$R_1 = C_0 R_0^{(n-1)/n}$$

and define C_1 in such a way that any $(n-1)$-convex function $u(x)$ satisfying the inequality

$$\|u-u_0'\|_{C[a+\Delta_1,b-\Delta_1]} \le R_1$$

satisfies the inequality

$$\|u'-u_0''\|_{C[a+\Delta_2,b-\Delta_2]} \le C_1\|u-u_0'\|^{(n-2)/(n-1)}_{C[a+\Delta_1,b-\Delta_1]}.$$

Then we put

$$R_2 = C_1 R_1^{(n-2)/(n-1)}$$

and continue the process.

Let an n-convex function $u(x)$ satisfy the inequality

$$\|u-u_0\|_{C[a,b]} \le R.$$

We have

$$\|u'-u_0'\|_{C[a+\Delta_1,b-\Delta_1]} \le C_0\|u-u_0\|^{(n-1)/n}_{C[a,b]} \le R_1.$$

Then we have

$$\|u''-u_0''\|_{C[a+\Delta_2,b-\Delta_2]} \le C_1\|u'-u_0'\|^{(n-2)/(n-1)}_{C[a+\Delta_1,b-\Delta_1]} \le C_1 C_0^{(n-2)/(n-1)}\|u-u_0\|^{(n-2)/n}_{C[a,b]} \le R_2.$$

And so on. Therefore we have

$$\|u^{(k)}-u_0^{(k)}\|_{C[a+\Delta,b-\Delta]} \le C\|u-u_0\|^{(n-k)/n}_{C[a,b]}$$

with some $C>0$. •

3. Estimation of C-norm by L-norm

Theorem 3 *Let $n \ge 1$. Let $u_0(x)$ be n-convex and C^n-differentiable on the interval $[a,b]$. For any $R>0$ and $0<\Delta<(b-a)/2$ there exists $C>0$ such that any n-convex function $u(x)$ from the ball*

$$\|u-u_0\|_{L_1[a,b]} \le R$$

satisfies the inequality

$$\|u-u_0\|_{C[a+\Delta,b-\Delta]} \le C\|u-u_0\|^{n/(n+1)}_{L_1[a,b]}.$$

Proof. Consider $(n+1)$-convex function

$$U_0(x) = \int_a^x u_0(t)\,dt.$$

By Theorem 2, there exists $C > 0$ such that any $(n+1)$-convex function $U(x)$ satisfying the inequality

$$\|U - U_0\|_{L_1[a,b]} \le R$$

satisfies the inequality

$$\|U' - U_0'\|_{C[a+\Delta,b-\Delta]} \le C\|U - U_0\|_{C[a,b]}^{n/(n+1)}.$$

Let an n-convex $u(x)$ satisfy the inequality

$$\|u - u_0\|_{L_1[a,b]} \le R.$$

We put

$$U(x) = \int_a^x u(t)\,dt.$$

Then

$$\begin{aligned}
\|U - U_0\|_{C[a,b]} &= \sup_{a\le x\le b}\left|\int_a^x u(t)\,dt - \int_a^x u_0(t)\,dt\right| \\
&\le \sup_{a\le x\le b}\int_a^x |u(t)\,dt - u_0(t)|\,dt \\
&\le \|u - u_0\|_{L_1[a,b]}.
\end{aligned}$$

Hence

$$\|U - U_0\|_{L_1[a,b]} \le R.$$

Consequently

$$\begin{aligned}
\|u - u_0\|_{C[a+\Delta,b-\Delta]} &= \|U' - U_0'\|_{C[a+\Delta,b-\Delta]} \\
&\le C\|U - U_0\|_{C[a,b]}^{n/(n+1)} \\
&\le C\|u - u_0\|_{L_1[a,b]}^{n/(n+1)}.
\end{aligned}$$

The theorem is proved. •

Bibliography

[1] S.Karlin, W.J.Studden. Tchebysheff systems: with applications in analysis and statistics. Interscience publications. Wiley & Sons.

[2] V.A.Malyshev. Nonlimear embedding theorems. St.Peterburg Math. J. Vol. 5 (1994), No. 6

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