Splines and Simplexes

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Abstract

Let $V \subset \mathbb{R}^n$ be a simplex. For any nonzero $w \in \mathbb{R}^n$ the function

$$B(t) = \frac{d}{\mathrm{vol} V} \frac{d}{dt} \mathrm{vol}\{x \in V : \ (w, x) \leq t\}$$

is a basis spline.

Introduction

In the paper we consider connections between polynomial splines and Euclidean simplexes. In the first and the second section we collect together necessary definitions and statements about interpolation polynomials and polynomial splines [1]. Geometric properties of basis splines are considered in the third section.

1. Hermite polynomials

Let $f(t)$ be a sufficiently smooth function, and let $t_0 < t_1 < \ldots < t_n$ be points of interpolation. The polynomial $L(t)$ of degree $\leq n$ is called the Lagrange polynomial if

$$L(t_i) = f(t_i)$$

for all $0 \leq i \leq n$.

2000 Mathematics Subject Classification. Primary 41-01, 65-01, 41A15, 65D07.

Key words and phrases. Polynomial splines, Euclidean simplexes.

Let

$$\omega_i(t) = \prod_{\substack{j=0 \\ j\neq i}}^{n} (t - t_j)$$

and

$$L_i(t) = \frac{\omega_i(t)}{\omega_i(t_i)}.$$

Then

$$L(t) = \sum_{i=0}^{n} f(t_i) L_i(t).$$

In particular,

$$F(t) = \sum_{i=0}^{n} F(t_i) L_i(t)$$

for any polynomial $F(t)$ of degree $\leq n$.

The function

$$\Delta_n f(t_0, \dots, t_n) = \sum_{i=0}^{n} \frac{f(t_i)}{\omega_i(t_i)}.$$

is called the divided difference of order n. We see that

$$\Delta_n f(t_0, \dots, t_n) = \frac{1}{n!} L^{(n)}(t).$$

We recall the well-known Newton formula

$$L(t) = \Delta_0 f(t_0) + \Delta_1 f(t_0, t_1)(t - t_0) + \dots + \Delta_n f(t_0, \dots, t_n)(t - t_0) \dots (t - t_{n-1}).$$

By the Newton formula, there exists a point $t_0 < \xi < t_n$ such that

$$\Delta_n f(t_0, \dots, t_n) = \frac{1}{n!} f^{(n)}(\xi).$$

Hence

$$\lim_{\varepsilon \to 0} \Delta_n f(t_0, t_0 + \varepsilon, \dots, t_0 + n\varepsilon) = \frac{1}{n!} f^{(n)}(t_0).$$

Since

$$\Delta_n f(t_0, t_0 + \varepsilon, \dots, t_0 + n\varepsilon) = \frac{1}{\varepsilon^n} \frac{1}{n!} \sum_{i=0}^{n} (-1)^{i+n} C_n^i f(t_0 + i\varepsilon),$$

we see that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \sum_{i=0}^{n} (-1)^{i+n} C_n^i f(t_0 + i\varepsilon) = f^{(n)}(t_0).$$

Let $t_0 < t_1 < \dots < t_m$ be points of interpolation with multiplicities $r_0 \geq 1, \dots, r_m \geq 1$. Let us put

$$n = r_0 + \dots + r_m - 1.$$

Then we see $n+1$ points in the sequence

$$\underbrace{t_0,\ldots,t_0}_{r_0},\ \underbrace{t_1,\ldots,t_1}_{r_1},\ \cdots,\ \underbrace{t_m,\ldots,t_m}_{r_m}.$$

The polynomial $H(t)$ of degree $\le n$ is called the Hermite polynomial if

$$H^{(p)}(t_i) = f^{(p)}(t_i)$$

for all $0 \le i \le m$ and $0 \le p \le r_i$.

Let

$$\Omega_i(t) = \prod_{\substack{j=0\\ j\neq i}}^{m} (t-t_j)^{r_j}$$

and

$$H_{ip}(t) = \frac{(t-t_i)^p}{p!}\Omega_i(t) \sum_{k=0}^{r_i-1-p} \left(\frac{1}{\Omega_i(t)}\right)^{(k)}\Bigg|_{t=t_i} \frac{(t-t_i)^k}{k!},$$

where $0 \le i \le m$ and $0 \le p \le r_i - 1$.

Theorem 1 *We have*

$$H_{ip}^{(q)}(t_j) = \begin{cases} 1 & \text{if} \quad j = i \text{ and } q = p, \\ 0 & \text{if} \quad j = i \text{ and } q \neq p, \\ 0 & \text{if} \quad j \neq i, \end{cases}$$

where $0 \le i, j \le m$, $0 \le p \le r_i - 1$ *and* $0 \le q \le r_j - 1$.

Proof. The identities

$$H_{ip}^{(q)}(t_j) = \begin{cases} 1 & \text{if} \quad j = i \text{ and } q = p, \\ 0 & \text{if} \quad j = i \text{ and } q \le p-1, \\ 0 & \text{if} \quad j \neq i \end{cases}$$

are obvious.

Let $u(t)$ be a differentiable functions such that $u(0) \neq 0$. Let us put

$$v(t) = \sum_{k=0}^{r} \left(\frac{1}{u(t)}\right)^{(k)}\Bigg|_{t=0} \frac{t^k}{k!}.$$

We claim that

$$(u(t)v(t))^{(q)}|_{t=0} = 0$$

for all $1 \le q \le r$. Indeed,

$$(u(t)v(t))^{(q)} = \sum_{k=0}^{q} C_q^k u^{(q-k)}(t) v^{(k)}(t).$$

Hence

$$(u(t)v(t))^{(q)}|_{t=0} = \sum_{k=0}^{q} C_q^k u^{(q-k)}(0) \left(\frac{1}{u(t)}\right)^{(k)}\Bigg|_{t=0} = \left(u(t)\frac{1}{u(t)}\right)^{(q)}\Bigg|_{t=0} = 1^{(q)} = 0.$$

By the statement, we have

$$H_{ip}^{(q)}(t_i) = C_q^p \left(\Omega_i(t) \sum_{k=0}^{r_i-1-p} \left(\frac{1}{\Omega_i(t)} \right)^{(k)} \Bigg|_{t=t_i} \frac{(t-t_i)^k}{k!} \right)^{(q-p)} \Bigg|_{t=t_i} = 0.$$

The theorem is proved. •

By Theorem 1, the Hermite polynomial has the form

$$H(t) = \sum_{i=0}^{m} \sum_{p=0}^{r_i} f^{(p)}(t_i) H_{ip}(t).$$

In particular,

$$F(t) = \sum_{i=0}^{m} \sum_{p=0}^{r_i} F^{(p)}(t_i) H_{ip}(t)$$

for any polynomial $F(t)$ of degree $\leq n$.

The function

$$\Delta_n f(\underbrace{t_0, \ldots, t_0}_{r_0}, \; \cdots, \; \underbrace{t_m, \ldots, t_m}_{r_m}) = \frac{1}{n!} H^{(n)}(t)$$

is called the divided difference of order n with multiple arguments. By definition, we put

$$\Delta_n f(t_0^{r_0}, \ldots, t_m^{r_m}) = \Delta_n f(\underbrace{t_0, \ldots, t_0}_{r_0}, \; \cdots, \; \underbrace{t_m, \ldots, t_m}_{r_m}).$$

It is clear that

$$\Delta_n f(t_0^{r_0}, \ldots, t_m^{r_m}) = \sum_{i=0}^{m} \frac{1}{(r_i-1)!} \sum_{p=0}^{r_i-1} C_{r_i-1}^p f^{(p)}(t_i) \left(\frac{1}{\Omega_i(t)} \right)^{(r_i-1-p)} \Bigg|_{t=t_i}.$$

The right-hand side may be written in the form

$$\Delta_n f(t_0^{r_0}, \ldots, t_m^{r_m}) = \sum_{i=0}^{n} \frac{1}{(r_i-1)!} \left(\frac{f(t)}{\Omega_i(t)} \right)^{(r_i-1)} \Bigg|_{t=t_i}.$$

Let the Lagrange polynomial $L^\varepsilon(t)$ be defined for the interpolation points

$$t_0, t_0+\varepsilon, \ldots, t_0+(r_0-1)\varepsilon, \; \ldots\ldots, \; t_m, t_m+\varepsilon, \ldots, t_m+(r_m-1)\varepsilon.$$

We put

$$t_{ip}^\varepsilon = t_i + p\varepsilon,$$

where $0 \leq i \leq m$ and $0 \leq p \leq r_i - 1$. Then

$$L^\varepsilon(t) = \sum_{i=0}^{m} \sum_{p=0}^{r_i-1} f(t_{ip}^\varepsilon) L_{ip}^\varepsilon(t)$$

with some polynomials $L_{ip}^\varepsilon(t)$.

Theorem 2 *We have*

$$\lim_{\varepsilon\to 0} L^\varepsilon(t) = H(t)$$

for any t.

Proof. By the identity

$$\sum_{p=0}^{r-1} x_p y_p = \sum_{p=0}^{r-1} \left(\sum_{k=0}^{p} (-1)^{p+k} C_p^k x_k \right) \left(\sum_{k=p}^{r-1} C_k^p y_k \right),$$

we write $L^\varepsilon(t)$ in the form

$$L^\varepsilon(t) = \sum_{i=0}^{m} \sum_{p=0}^{r_i-1} \left(\frac{1}{\varepsilon^p} \sum_{k=0}^{p} (-1)^{p+k} C_p^k f(t_{ik}^\varepsilon) \right) \left(\varepsilon^p \sum_{k=p}^{r_i-1} C_k^p L_{ik}^\varepsilon(t) \right).$$

Let

$$H_{ip}^\varepsilon(t) = \varepsilon^p \sum_{k=p}^{r_i-1} C_k^p L_{ik}^\varepsilon(t).$$

We claim that

$$\lim_{\varepsilon\to 0} H_{ip}^\varepsilon(t) = H_{ip}(t).$$

Indeed, we put $F(t) = H_{ip}(t)$ and obtain

$$H_{ip}(t) = \sum_{j=0}^{n} \sum_{q=0}^{r_j-1} \left(\frac{1}{\varepsilon^q} \sum_{k=0}^{q} (-1)^{q+k} C_q^k H_{ip}(t_{jk}^\varepsilon) \right) H_{jq}^\varepsilon(t),$$

where $0 \le i \le m$ and $0 \le p \le r_i - 1$. Consider the vector h with elements

$$h_{ip} = H_{ip}(t),$$

the vector h^ε with elements

$$h_{jq}^\varepsilon = H_{jq}^\varepsilon(t),$$

and the matrix A^ε with elements

$$A_{(ip),(jq)}^\varepsilon = \frac{1}{\varepsilon^q} \sum_{k=0}^{q} (-1)^{q+k} C_q^k H_{ip}(t_{jk}^\varepsilon).$$

Then

$$h = A^\varepsilon h^\varepsilon.$$

We see that

$$\lim_{\varepsilon\to 0} A^\varepsilon = E,$$

where E is the identity matrix. Therefore,

$$\lim_{\varepsilon\to 0} h^\varepsilon = h.$$

Hence

$$\lim_{\varepsilon\to 0} H^{\varepsilon}_{ip}(t) = H_{ip}(t).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^p}\sum_{k=0}^{p}(-1)^{p+k}C_p^k f(t^{\varepsilon}_{ik}) = f^{(p)}(t_i),$$

we have

$$\lim_{\varepsilon\to 0} L^{\varepsilon}(t) = H(t).$$

The theorem is proved. •

Corollary. We have

$$\Delta_n f(t_0^{r_0},\ldots,t_m^{r_m}) = \lim_{\varepsilon\to 0}\Delta_n f(t_0, t_0+\varepsilon,\ldots,t_0+(r_0-1)\varepsilon,\ldots,t_m,t_m+\varepsilon,\ldots,t_m+(r_m-1)\varepsilon).$$

2. Basis splines

Let

$$t_+^k = \begin{cases} t^k & \text{if} \quad t\geq 0,\\ 0 & \text{if} \quad t<0.\end{cases}$$

A function

$$f(t) = \sum_{i=0}^{m} c_i(t-t_i)_+^{k_i}$$

is called a polynomial spline.

Theorem 3 *Let $a_1 < \ldots < a_m$, $k_1\geq 1,\ldots,k_m\geq 1$, and $n = k_1+\cdots+k_m$. The family*

$$(t+a_1)^{n-1},\ldots,(t+a_1)^{n-k_1},\ \ldots\ldots,\ (t+a_m)^{n-1},\ldots,(t+a_m)^{n-k_m},$$

is a basis in the space of polynomials of degree $\leq n-1$.

Proof. Let $k_0 = 0$, and let

$$f_{k_0+\cdots+k_{i-1}+j}(t) = (t+a_i)^{n-j},$$

where $1\leq i\leq m$ and $1\leq j\leq k_i$. It suffices to check that

$$\det\{f_j(x_i)\}_{i,j=1}^n \neq 0,$$

where $x_1 < x_2 < \ldots < x_n$. Let

$$y_{k_0+\cdots+k_{i-1}+j} = a_i+\varepsilon(j-1),$$

where $1\leq i\leq m$ and $1\leq j\leq k_i$. We put

$$D(\varepsilon) = \det\{(x_i+y_j)^{n-1}\}_{i,j=1}^n.$$

It is easy to see that

$$D(\varepsilon)=\prod_{i=0}^{n-1}C_{n-1}^{i}\prod_{0\le i<j\le n-1}(x_i-x_j)\prod_{0\le i<j\le n-1}(y_j-y_i).$$

Let

$$N=\frac{(k_1-1)k_1}{2}+\cdots+\frac{(k_m-1)k_m}{2}.$$

On the one hand,

$$D(\varepsilon)=\varepsilon^N\prod_{i=0}^{n-1}C_{n-1}^{i}\prod_{i=1}^{m}\prod_{j=1}^{k_i-1}\frac{(k_i-1)!}{j!}\prod_{0\le i<j\le n-1}(x_i-x_j)\prod_{1\le i<j\le m}(a_j-a_i)^{k_ik_j}+o(\varepsilon^N).$$

On the other hand,

$$D(\varepsilon)=\varepsilon^N\prod_{i=1}^{m}\prod_{j=1}^{k_i-1}\frac{(n-1)!}{(n-1-j)!}\det\{f_j(x_i)\}_{i,j=1}^{n}+o(\varepsilon^N).$$

This means that

$$\det\{f_j(x_i)\}_{i,j=1}^{n}=C\prod_{0\le i<j\le n-1}(x_i-x_j)\prod_{1\le i<j\le m}(a_j-a_i)^{k_ik_j},$$

where

$$C=\prod_{i=0}^{n-1}C_{n-1}^{i}\prod_{i=1}^{m}\prod_{j=1}^{k_i-1}\frac{(n-1-j)!(k_i-1)!}{(n-1)!j!}.$$

Therefore,

$$\det\{f_j(x_i)\}_{i,j=1}^{n}\neq 0.$$

This proves the theorem. •

Theorem 4 *Let $t_0<t_1<\ldots<t_m$, $r_0\ge 1, r_1\ge 1,\ldots,r_m\ge 1$, and $n=r_0+r_1+\cdots+r_m-1$. There exists a unique spline*

$$B(t)=\sum_{i=0}^{m}\sum_{p=0}^{r_i-1}b_{ip}(t-t_i)_+^{n-1-p}$$

such that

$$\int_{-\infty}^{\infty}B(t)\,dt=1$$

and

$$\begin{array}{lll}B(t)=0 & \text{if} & t\le t_0,\\ B(t)>0 & \text{if} & t_0<t<t_m,\\ B(t)=0 & \text{if} & t\ge t_m.\end{array}$$

Proof. We claim that

$$B(t) = (-1)^n n \Delta_n (t-\tau)_+^{n-1}(t_0^{r_0}, \ldots, t_n^{r_n}),$$

where the divided difference is taken with respect to τ. In particular, we have

$$b_{ip} = (-1)^{n+p} n C_{n-1}^p \frac{1}{(r_i-1-p)!} \left(\frac{1}{\omega_i(\tau)} \right)^{(r_i-1-p)} \Bigg|_{\tau=t_i},$$

where $0 \le i \le m$ and $0 \le p \le r_i - 1$. Here

$$\omega_i(\tau) = \prod_{\substack{j=0 \\ j\neq i}}^{m} (\tau - t_j)^{k_j},$$

where $0 \le i \le m$.

Let $r_0 = \ldots = r_m = 1$. In this case $m = n$ and we have to prove that

$$B(t) = \sum_{i=0}^{n} b_i (t-t_i)_+^{n-1},$$

where

$$b_i = n \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i}.$$

Obviously

$$B(t) = 0$$

for $t \le t_0$. Differentiating n times

$$\sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t_j - \tau}{t_j - t_i} (t-t_i)^{n-1} = (t-\tau)^{n-1}$$

with respect to τ, we get

$$B(t) = 0$$

for $t \ge t_n$. Differentiating n times

$$(t_n - \tau)^n = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t_j - \tau}{t_j - t_i} (t_n - t_i)^n,$$

with respect to τ, we get

$$\sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i} (t_n - t_i)^n = 1.$$

Hence

$$\int_{-\infty}^{\infty} B(t)\,dt = \sum_{i=0}^{n} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{1}{t_j - t_i}(t_n - t_i)^n = 1.$$

Let us check that

$$b_0 + \cdots + b_k \neq 0$$

for all $0 \leq k \leq n-1$. Assume that $b_0 + \cdots + b_k = 0$ for some $0 \leq k \leq n-1$. Consider the polynomial

$$f(\tau) = \sum_{i=0}^{k} \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{t_j - \tau}{t_j - t_i}$$

of degree $\leq n-1$. We see that

$$f(t_0) = \ldots = f(t_k) = 1$$

and

$$f(t_{k+1}) = \ldots = f(t_n) = 0.$$

Therefore $f'(\tau)$ has $n-1$ mutually different roots. Hence $f'(\tau) \equiv 0$. Therefore $f(\tau) \equiv \text{const}$. The contradiction proves the inequality $b_0 + \cdots + b_k \neq 0$.

Let us check that

$$B(t) \neq 0$$

for all $t_0 < t < t_n$. Assume that $B(t_*) = 0$ for some $t_0 < t_* < t_n$. By the definition of $B(t)$, we see that

$$B^{(i)}(t_0) = B^{(i)}(t_n) = 0$$

for all $0 \leq i \leq n-2$. Hence $B^{(n-2)}(t)$ has $n+1$ mutually different roots between t_0 and t_n. Since $B^{(n-2)}(t)$ is peace-wise linear with knots $t_0 < \ldots < t_n$, we have $B^{(n-2)}(t) \equiv 0$ on $[t_k, t_{k+1}]$ with some $0 \leq k \leq n-1$. On the interval $[t_k, t_{k+1}]$ the basis spline $B(t)$ is equal to a polynomial of degree $\leq n-3$. Hence $b_0 + \ldots + b_k = 0$. The contradiction proves that $B(t) \neq 0$ for all $t_0 < t < t_n$.

Since

$$\int_{-\infty}^{\infty} B(t)\,dt = 1,$$

we have

$$B(t) > 0$$

for all $t_0 < t < t_n$. This prove the existence of $B(t)$.

Consider a spline

$$C(t) = \sum_{i=0}^{n} c_i (t - t_i)_+^{n-1}$$

such that

$$\int_{-\infty}^{\infty} C(t)\,dt = 1$$

and

$$C(t) = 0$$

for all $t \geq t_n$. The system of polynomials

$$(t - t_1)^{n-1}, \ldots, (t - t_n)^{n-1}$$

is a basis in the space of polynomials of degree $\leq n - 1$. By identities

$$b_0(t - t_0)^{n-1} = -\sum_{i=1}^{n} b_i(t - t_i)^{n-1}$$

and

$$c_0(t - t_0)^{n-1} = -\sum_{i=1}^{n} c_i(t - t_i)^{n-1},$$

we conclude that

$$c_i = \frac{c_0}{b_0} b_i$$

for all $i = 0, 1, \ldots, n$. Hence

$$C(t) = \frac{c_0}{b_0} B(t).$$

By integration, we obtain $c_0 = b_0$. Hence $c_i = b_i$ for all $i = 0, 1, \ldots$. This proves the uniqueness of $B(t)$.

Let $r_0 \geq 1, r_1 \geq 1, \ldots, r_m \geq 1$ and $n = r_0 + r_1 + \cdots + r_m - 1$.

Let us assume that $n - r_i = 1$ for some $0 \leq i \leq m$. Then $m = 1$. Therefore $r_0 = 1$ and $r_1 = n$ or $r_0 = n$ and $r_1 = 1$. If $r_0 = 1$ and $r_1 = n$, then

$$B(t) = \frac{n}{(t_1 - t_0)^n}(t - t_0)_+^{n-1} - \sum_{p=0}^{n-1} C_{n-1}^p \frac{n}{(t_1 - t_0)^{n-p}}(t - t_1)_+^{n-1-p}.$$

The case $r_0 = n$ and $r_1 = 1$ is considered analogously.

Let us assume that $n - r_i \geq 1$ for all $0 \leq i \leq m$. We put

$$B(t) = (-1)^n n \Delta_n (t - \tau)_+^{n-1}(t_0^{r_0}, \ldots, t_m^{r_m})$$

and

$$B^\varepsilon(t) = (-1)^n n \Delta_n (t - \tau)_+^{n-1}(t_0, t_0 + \varepsilon, \ldots, t_0 + (r_0 - 1)\varepsilon, \ldots, t_m, t_m + \varepsilon, \ldots, t_m + (r_m - 1)\varepsilon),$$

where

$$t_{ip}^\varepsilon = t_i + p\varepsilon$$

for $0 \leq i \leq m$ and $0 \leq p \leq r_i - 1$. We have

$$B(t) = \lim_{\varepsilon \to 0} B^\varepsilon(t).$$

By continuity the spline $B(t)$ satisfies all the demand conditions. This prove the existence of $B(t)$.

By Theorem 3, we obtain the following statements. If $r_0 \geq 2$, then the family

$$(t-t_0)^{n-1}, \ldots, (t-t_0)^{n-r_0+1}; (t-t_1)^{n-1}, \ldots, (t-t_1)^{n-r_1}; \ \ldots; \ (t-t_m)^{n-1}, \ldots, (t-t_m)^{n-r_m}$$

is a basis in the space of polynomials of degree $\leq n-1$. If $r_0 = 1$, then the family

$$(t-t_1)^{n-1}, \ldots, (t-t_1)^{n-r_1}; \ \ldots; \ (t-t_m)^{n-1}, \ldots, (t-t_m)^{n-r_m}$$

is a basis in the space of polynomials of degree $\leq n-1$. This proves the uniqueness of $B(t)$. $\bullet$

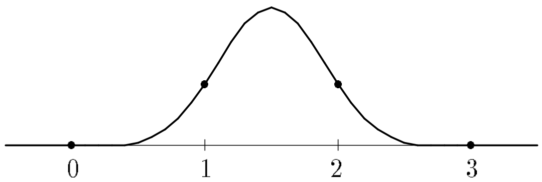
The spline $B(t)$ constructed in Theorem 4 is called the basis spline. For example, if $m = 3$, $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3$ and $r_0 = r_1 = r_2 = r_3 = 2$, then $B(t)$ has coefficients

$$\begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \\ b_{30} & b_{31} \end{bmatrix} = \begin{bmatrix} -77/108 & 7/6 \\ -7/4 & 21/2 \\ 7/4 & 21/2 \\ 77/108 & 7/6 \end{bmatrix}.$$

The spline

$$B(t) = b_{00}t_+^6 + b_{01}t_+^5 + b_{10}(t-1)_+^6 + b_{11}(t-1)_+^5 + b_{20}(t-2)_+^6 + b_{21}(t-2)_+^5 + b_{30}(t-3)_+^6 + b_{31}(t-3)_+^5$$

we see on the picture.



3. Splines and volumes

Let $\subset \mathbb{R}^n$ be a Euclidean simplex with vertexes $A_0, A_1, \ldots, A_n$. Consider a nonzero direction $w \in \mathbb{R}^n$. Let us assume that

$$(w, A_i) \leq (w, A_j)$$

for all $0 \leq i \leq j \leq n$. If $t_* = (w, A_i)$ for some $0 \leq i \leq n$, then we say that t_* is a critical level of the direction w. Let

$$t_0 < t_1 < \ldots < t_m$$

be all critical levels of the direction w. By r_i we denote the number of vertexes on the critical level t_i. We note that $r_0 + r_1 + \cdots + r_m = n + 1$.

Let

$$\Gamma_t = \{x \in V : \ (w, x) = t\}$$

and

$$V_t = \{x \in V : \ (w, x) \leq t\}.$$

We see that

$$\mathrm{vol}\Gamma_t = \frac{d}{dt}\mathrm{vol}V_t.$$

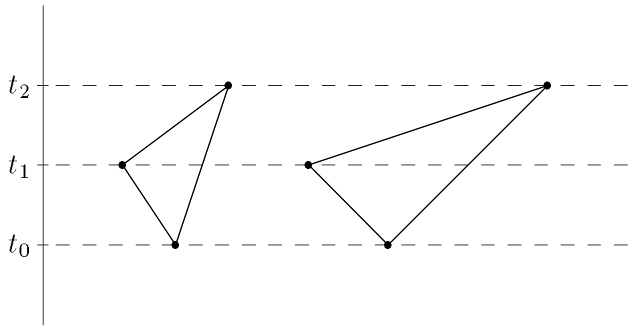
Therefore,

$$F(t) = \frac{\mathrm{vol}V_t}{\mathrm{vol}V}$$

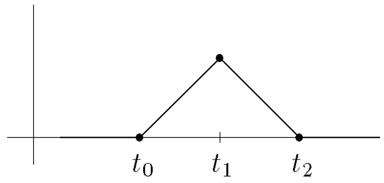
is the distribution function and

$$f(t) = \frac{\mathrm{vol}\Gamma_t}{\mathrm{vol}V}$$

is the density of $F(t)$. We claim that $F(t)$ is uniquely defined by $t_0, t_1, \ldots, t_m$ and $r_0, r_1, \ldots, r_m$. In the case $n = 2$ the fact is obvious. Indeed, the triangles



have equal distribution functions with the density of the form.



Theorem 5 *Let the basis spline $B(t)$ be defined for $t_0, t_1, \ldots, t_m$ and $r_0, r_1, \ldots, r_m$. Then $f(t) = B(t)$.*

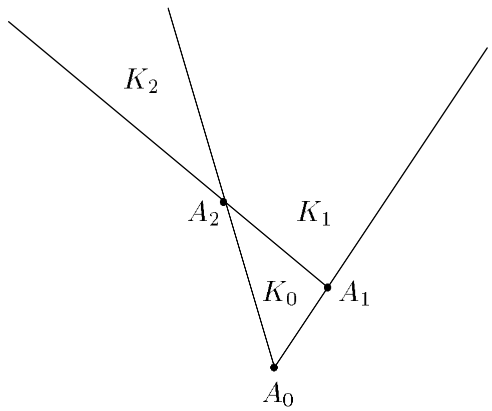
Proof. We first consider the special case $r_0 = \ldots = r_m = 1$. Note that $m = n$. Consider cones

$$K^i = A_i + \mathrm{co}\{A_i - A_0,\ A_i - A_1,\ \ldots,\ A_i - A_{i-1},\ A_{i+1} - A_i,\ A_{i+2} - A_i,\ \ldots, A_n - A_i\},$$

where $i = 0, 1, \ldots, n$. In the case $n = 2$ cones

$$\begin{aligned} K_0 &= A_0 + \mathrm{co}\{A_1 - A_0,\ A_2 - A_0\} \\ K_1 &= A_1 + \mathrm{co}\{A_1 - A_0,\ A_2 - A_1\} \\ K_2 &= A_2 + \mathrm{co}\{A_2 - A_0,\ A_2 - A_1\} \end{aligned}$$

are shown on the picture.



Let $x \in K_i$. Then for uniquely defined

$$\lambda_0 \geq 0, \dots, \lambda_{i-1} \geq 0, \lambda_{i+1} \geq 0, \dots, \lambda_n \geq 0$$

we have

$$x = A_i + \lambda_0(A_i - A_0) + \dots + \lambda_{i-1}(A_i - A_{i-1}) + \lambda_{i+1}(A_{i+1} - A_i) + \dots + \lambda_n(A_n - A_i).$$

Hence

$$x = A_0 - (\lambda_1 v_1 + \dots + \lambda_{i-1} v_{i-1}) + (\lambda_0 + \dots + \lambda_{i-1} + 1 - \lambda_{i+1} - \dots - \lambda_n) v_i + \lambda_{i+1} v_{i+1} + \dots + \lambda_n v_n,$$

where

$$v_1 = A_1 - A_0, \dots, v_n = A_n - A_0.$$

Let us check that

$$\mathrm{Int} V = \mathrm{Int} K_0 \setminus K_1.$$

The inclusion $\mathrm{Int} V \subset \mathrm{Int} K_0$ is obvious. Let $x \in \mathrm{Int} K_0$ and $x \notin \mathrm{Int} V$. Then for some $\lambda_1 > 0, \dots, \lambda_n > 0$ we have

$$x = A_0 + \lambda_1 v_1 + \dots + \lambda_n v_n$$

and

$$\lambda_1 + \dots + \lambda_n \geq 1.$$

Put

$$\lambda_0 = \lambda_1 + \dots + \lambda_n - 1.$$

From

$$x = A_0 + (\lambda_0 + 1 - \lambda_2 - \dots - \lambda_n) v_1 + \dots + \lambda_n v_n$$

it follows that $x \in K_1$.

Let $0 \leq i \leq n-2$. Let us check that

$$\mathrm{Int} K_i \cap \mathrm{Int} K_{i+2} = \emptyset.$$

Let $x \in \mathrm{Int} K_i$. Then for some

$$\lambda_0 > 0, \dots, \lambda_{i-1} > 0, \lambda_{i+1} > 0, \dots, \lambda_n > 0$$

we have

$$x = A_0 - (\lambda_1 v_1 + \dots + \lambda_{i-1} v_{i-1}) + (\lambda_0 + \dots + \lambda_{i-1} + 1 - \lambda_{i+1} - \dots - \lambda_n) v_i + \lambda_{i+1} v_{i+1} + \dots + \lambda_n v_n.$$

Let $y \in \mathrm{Int} K_{i+2}$. Then for some

$$\mu_0 > 0, \dots, \mu_{i+1} > 0, \mu_{i+3} > 0, \dots, \mu_n > 0$$

we have

$$y = A_0 - (\mu_1 v_1 + \dots + \mu_{i+1} v_{i+1}) + (\mu_0 + \dots + \mu_{i+1} + 1 - \mu_{i+3} - \dots - \mu_n) v_{i+2} + \mu_{i+3} v_{i+3} + \dots + \mu_n v_n.$$

Since $\lambda_{i+1} \neq -\mu_{i+1}$, we have $x \neq y$.

Let $0 \leq i \leq n-2$. Let us check that

$$K_{i+1} \subset K_i \cup K_{i+2}.$$

Let $x \in K_{i+1}$. Then for some

$$\lambda_0 \geq 0, \dots, \lambda_i \geq 0, \lambda_{i+2} \geq 0, \dots, \lambda_n \geq 0$$

we have

$$x = A_0 - (\lambda_1 v_1 + \dots + \lambda_i v_i) + (\lambda_0 + \dots + \lambda_i + 1 - \lambda_{i+2} - \dots - \lambda_n) v_{i+1} + \lambda_{i+2} v_{i+2} + \dots + \lambda_n v_n.$$

We see that

$$\lambda_0 + \dots + \lambda_i + 1 - \lambda_{i+2} - \dots - \lambda_n \geq 0$$

implies $x \in K_{i-1}$ and

$$\lambda_0 + \dots + \lambda_i + 1 - \lambda_{i+2} - \dots - \lambda_n \leq 0$$

implies $x \in K_{i+1}$.

Consider the set

$$K_i^t = \{x \in K^i : (w, x) \leq t\}.$$

If $t < t_i$, then $K_i^t = \emptyset$. If $t = t_i$, then $K_i^t = A_i$. If $t > t_i$ then K_i^t is a simplex. The simplex K_i^t has one vertex at the points A_i and n vertexes at the points

$$A_i + \frac{t - t_i}{t_j - t_i}(A_j - A_i),$$

where $1 \leq j \leq n$ and $j \neq i$. These n vertexes belong to the plane

$$(w, x) = t.$$

Therefore,

$$\mathrm{vol} K_i^t = (-1)^i \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i} (t - t_i)_+^n \,\mathrm{vol} V.$$

Let $K_{n+1}^t = \emptyset$. Then

$$\mathrm{vol} K_i^t = \mathrm{vol}(K_{i-1}^t \cap K_i^t) + \mathrm{vol}(K_i^t \cap K_{i+1}^t)$$

for all $i = 1, \ldots, n$. From

$$\mathrm{vol} V_t = \mathrm{vol} K_0^t - \mathrm{vol}(K_0^t \cap K_1^t)$$

we have

$$\mathrm{vol} V_t = \mathrm{vol} K_0^t - \mathrm{vol} K_1^t + \ldots + (-1)^n \mathrm{vol} K_n^t.$$

Therefore,

$$\mathrm{vol} V_t = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i} (t - t_i)_+^n \,\mathrm{vol} V.$$

This proves the identity

$$F(t) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i} (t - t_i)_+^n.$$

From the identity it follows that

$$f(t) = n \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{t_j - t_i} (t - t_i)_+^{n-1} = B(t).$$

Consider the general case $r_0 \geq 1, \ldots, r_m \geq 1$. Let $r_{-1} = 0$. Put

$$A^\varepsilon_{r_{-1}+\ldots+r_{i-1}+p} = A_{r_{-1}+\ldots+r_{i-1}+p} + p\varepsilon\omega,$$

where $0 \leq i \leq m$ and $0 \leq p \leq r_i - 1$. Here $\varepsilon > 0$ is sufficiently small. Denote by V^ε the simplex with vertexes $A_0^\varepsilon, \ldots, A_n^\varepsilon$. The simplex V^ε has the following critical levels

$$t^\varepsilon_{r_{-1}+\ldots+r_{i-1}+p} = t_{r_{-1}+\ldots+r_{i-1}+p} + p\varepsilon,$$

where $0 \leq i \leq m$ and $0 \leq p \leq r_i - 1$. Let the basis spline $B^\varepsilon(t)$ be define for $t_0^\varepsilon, \ldots, t_n^\varepsilon$. Then

$$f^\varepsilon(t) = B^\varepsilon(t),$$

where $f^\varepsilon(t)$ corresponds to V^ε.

Let $t \neq t_i$ for all $0 \leq i \leq m$. By continuity,

$$\lim_{\varepsilon \to 0} f^\varepsilon(t) = f(t).$$

Since

$$\lim_{\varepsilon \to 0} B^\varepsilon(t) = B(t),$$

we obtain $f(t) = B(t)$. The theorem is proved. $\bullet$

Corollary. We have

$$F(t) = \int\limits_{-\infty}^{t} B(\tau)\, d\tau.$$

Example. Let us calculate the distribution function

$$F(t) = \frac{1}{2^n}\mathrm{vol}\left\{x \in [-1,1]^n : \ \frac{x_1 + \cdots + x_n}{n} \le t\right\}.$$

Consider the direction

$$w = \frac{1}{n}(1, \ldots, 1).$$

and the simplex V with the vertexes

$$A_i = (\underbrace{-1, \ldots, -1}_{n-i}; \underbrace{1, \ldots, 1}_{i}),$$

where $0 \le i \le n$. The points

$$-1,\ -1 + 2\frac{1}{n},\ -1 + 2\frac{2}{n},\ \ldots,\ -1 + 2\frac{n-1}{n},\ 1$$

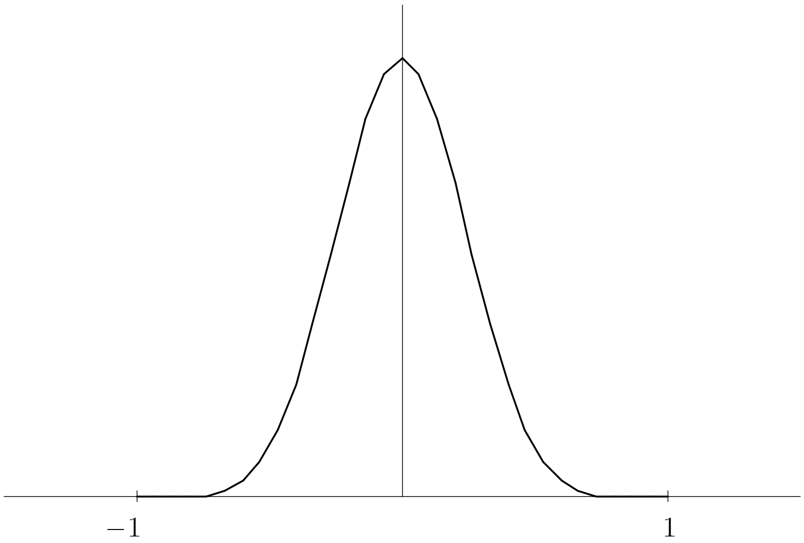
are the critical levels for V and w. Therefore,

$$F(t) = \frac{1}{n!}\left(\frac{n}{2}\right)^n \sum_{i=0}^{n} (-1)^i C_n^i \left(t + 1 - 2\frac{i}{n}\right)_+^n$$

and

$$f(t) = \frac{1}{(n-1)!}\left(\frac{n}{2}\right)^n \sum_{i=0}^{n} (-1)^i C_n^i \left(t + 1 - 2\frac{i}{n}\right)_+^{n-1}.$$

For $n = 6$ the density $f(t)$ is shown on the picture



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