Triangulations with Regular Structures

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Abstract

The direct product of two simplices and the intersection of a simplex with a half-space have triangulations with regular structure. This gives natural triangulations for convex polytopes and permits to construct some nontrivial homeomorphisms.

Contents

Introduction

In the paper we consider polytopes with regular triangulations. The main examples are the direct product of two simplices and the intersection of a simplex with a half-space. In particular, the last triangulation permits to obtain a natural triangulation for any convex polytope. Indeed, let a convex polytope belong to a simplex. Since the polytope may be represented as an intersection of half-spaces, the triangulation of the polytope is constructed step by step. At each step we use the triangulation of the intersection of a simplex with a half-space.

The triangulations with regular structure permit to construct some homeomorphisms. Let

$$\Delta^n = \{\lambda \in \mathbb{R}^{n+1} :\ \lambda_0 \geq 0, \dots, \lambda_n \geq 0,\ \lambda_0 + \cdots + \lambda_n = 1\}$$

be the unite n-dimensional simplex. We assume that $\Delta^{-1} = \emptyset$. Consider a simplicial surjection

$$F :\ \Delta^n \longrightarrow \Delta^k.$$

Then the cylinder $\mathrm{Cyl}F$ of the mapping F is homeomorphic to the ball

$$D^{n+1} \approx \mathrm{cyl}\Delta^k * \Delta^{n-k-1}$$

and has a natural triangulation with $k+1$ simplices of dimension $n+1$.

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1. Associated triagulations

Let $A_0, A_1, \ldots, A_n$ be a set of points in the space $\mathbb{R}^m$. By

$$\Delta = \operatorname{conv}\{A_0, A_1, \ldots, A_n\}$$

we denote the convex hull of the set $A_0, A_1, \ldots, A_n$. The set $A_0, A_1, \ldots, A_n$ is called nonsingular if

$$\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$$

whenever

$$\lambda_0 + \lambda_1 + \cdots + \lambda_n = 0$$

and

$$\lambda_0 A_0 + \lambda_1 A_1 + \cdots + \lambda_n A_n = 0.$$

If $A_0, A_1, \ldots, A_n$ is nonsingular, then Δ is a simplex of dimension n.

Let $T = \{\Delta\}$ be a finite collection of simplices $\Delta \subset \mathbb{R}^m$. The collection T is called a triangulation [1] if the following conditions hold:

1^0. If Γ is a face of Δ and $\Delta \in T$, then $\Gamma \in T$.

2^0. If $\Gamma, \Delta \in T$ and $\Gamma \subset \Delta$, then Γ is a face of Δ.

3^0. If $\Gamma, \Delta \in T$, then $\Gamma \cap \Delta \in T$.

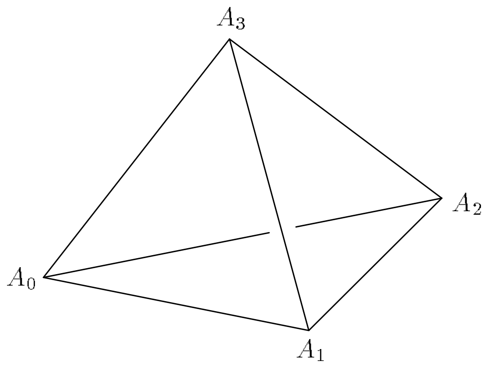
Example: The collection of all faces of a simplex Δ is a triangulation.

Let G be an oriented graph without closed paths. Let each vertex $v \in G$ be associated with a point $A_v \in \mathbb{R}^m$, and let the set $A_{v_0}, A_{v_1}, \ldots, A_{v_n}$ be nonsingular for any path $v_0, v_1, \ldots, v_n$ on the graph G. If simplices

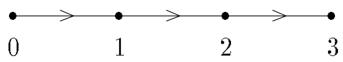
$$\Delta = \operatorname{conv}\{A_{v_0}, A_{v_1}, \ldots, A_{v_n}\}$$

with their faces form a triangulation, then we say that the triangulation is associated with the graph G. Obviously, for construction of the associated triangulation we may use only maximal paths on the graph G.

Example: The standard triagulation of the simplex



is associated with the graph



2. Direct products

All graphs that we use for construction of associated triangulations belong to the unit lattice. In particular, vertices and edges of our graphs are vertices and edges of the lattice. Since edges of the lattice are oriented in positive directions, we do not write arrows on the edges of our graphs.

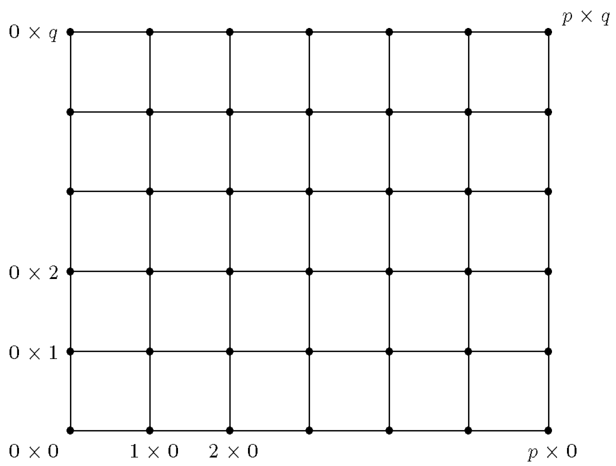
In the section we consider the standard triangulation of the direct product of two simplices. Let $A_0, \dots, A_p \in \mathbb{R}^m$ and $B_0, \dots, B_q \in \mathbb{R}^l$ be nonsingular sets. Consider simplices

$$X_p = \operatorname{conv}\{A_0, \dots, A_p\}$$

and

$$Y_q = \operatorname{conv}\{B_0, \dots, B_q\}.$$

Let S_{pq} be the graph



We see that S_{pq} has $C^p_{p+q} = C^q_{p+q}$ maximal paths connecting 0×0 and $p \times q$.

Assume that the vertex $i \times j \in S_{pq}$ is associated with the point $A_i \times B_j \in X_p \times Y_q$. We claim that the triangulation associated with the graph S_{pq} is the triangulation of the direct product $X_p \times Y_q$. In particular, the triangulation has $C^p_{p+q} = C^q_{p+q}$ simplices of maximal dimension $n = p + q$.

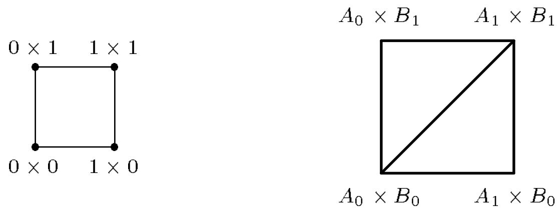
Examples. If $p = 1$ and $q = 1$, then S_{11} has two maximal paths

$$\begin{matrix} 0 \times 0 & 1 \times 0 & 1 \times 1 \\ 0 \times 0 & 0 \times 1 & 1 \times 1 \end{matrix}$$

Hence the triangulation of $X_1 \times Y_1$ has two maximal simplices with vertices

$$\begin{matrix} A_0 \times B_0 & A_1 \times B_0 & A_1 \times B_1 \\ A_0 \times B_0 & A_0 \times B_1 & A_1 \times B_1 \end{matrix}$$

Picture:



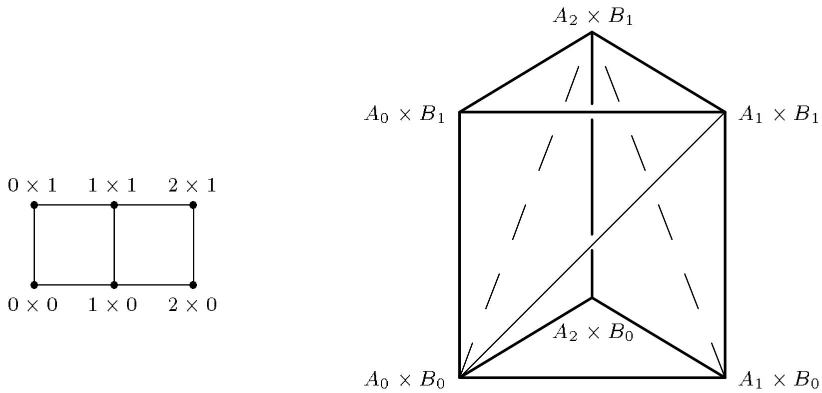
If $p = 2$ and $q = 1$, then $S_{2\,1}$ has three maximal paths

$$\begin{array}{cccc} 0\times 0 & 1\times 0 & 2\times 0 & 2\times 1 \\ 0\times 0 & 1\times 0 & 1\times 1 & 2\times 1 \\ 0\times 0 & 0\times 1 & 1\times 1 & 2\times 1 \end{array}$$

Hence the triangulation of $X_2 \times Y_1$ has three maximal simplices with vertices

$$\begin{array}{cccc} A_0\times B_0 & A_1\times B_0 & A_2\times B_0 & A_2\times B_1 \\ A_0\times B_0 & A_1\times B_0 & A_1\times B_1 & A_2\times B_1 \\ A_0\times B_0 & A_0\times B_1 & A_1\times B_1 & A_2\times B_1 \end{array}$$

Picture:



Theorem 1 *The triangulation associated with S_{pq} is the triangulation of $X_p \times Y_q$.*

Proof. Part 1. Let

$$i_0 \times j_0,\ i_1 \times j_1,\ \ldots,\ i_n \times j_n$$

be a maximal path. We claim that the set

$$A_{i_0} \times B_{j_0}, A_{i_1} \times B_{j_1}, \ldots, A_{i_n} \times B_{j_n}$$

is nonsingular.

We see that

$$0 = i_0 \le i_1 \le \cdots \le i_n = p$$

$$0 = j_0 \le j_1 \le \cdots \le j_n = q$$

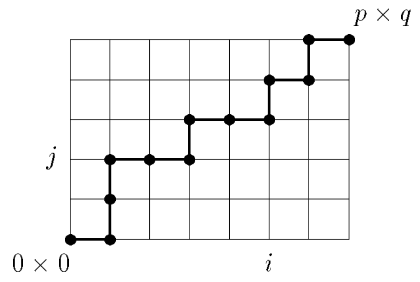
and

$$i_k + j_k = k$$

for all $0 \le k \le n$. Put

$$\begin{aligned} I(i) &= \{k : \ i_k = i\}, \\ J(j) &= \{k : \ j_k = j\}, \end{aligned}$$

where $0 \le i \le p$ and $0 \le j \le q$.



Assume that

$$\lambda_0 + \lambda_1 + \cdots + \lambda_n = 0$$

and

$$\lambda_0 A_{i_0} \times B_{j_0} + \lambda_1 A_{i_1} \times B_{j_1} + \cdots + \lambda_n A_{i_n} \times B_{j_n} = 0.$$

Since $A_0, A_1, \ldots, A_p$ and $B_0, B_1, \ldots, B_q$ are nonsingular, we have

$$\sum_{k \in I(i)} \lambda_k = 0$$

for $0 \le i \le p$ and

$$\sum_{k \in J(j)} \lambda_k = 0$$

for $0 \le j \le q$.

It is easy to check that

$$\max I(i) \ne \max J(j)$$

for $0 \le i \le p-1$ and $0 \le j \le q$, that

$$\max I(i') < \max I(i'')$$

for $0 \le i' < i'' \le p$, and that

$$\max J(j') < \max J(j'')$$

for $0 \le j' < j'' \le q$. This means that $\lambda_0, \lambda_1, \ldots, \lambda_n$ satisfy a linear system with a matrix of rank $n+1$. Therefore,

$$\lambda_0 = \lambda_1 = \ldots = \lambda_n = 0.$$

By definition, the set

$$A_{i_0} \times B_{j_0}, A_{i_1} \times B_{j_1}, \ldots, A_{i_n} \times B_{j_n}$$

is nonsingular.

Part 2. We claim that for any $x \times y \in X_p \times Y_q$ there exists a maximal path

$$i_0 \times j_0,\ i_1 \times j_1,\ \ldots,\ i_n \times j_n$$

such that

$$x \times y \in \mathrm{conv}\{A_{i_0} \times B_{j_0}, A_{i_1} \times B_{j_1}, \ldots, A_{i_n} \times B_{j_n}\}.$$

Let

$$x = \lambda_0 A_0 + \cdots + \lambda_p A_p$$

and

$$y = \mu_0 B_0 + \cdots + \mu_q B_q.$$

Put

$$\begin{array}{lcl} \alpha_0 & = & \lambda_0 \\ \alpha_1 & = & \lambda_0 + \lambda_1 \\ & \cdots & \\ \alpha_{p-1} & = & \lambda_0 + \lambda_1 + \cdots + \lambda_{p-1} \\ \beta_0 & = & \mu_0 \\ \beta_1 & = & \mu_0 + \mu_1 \\ & \cdots & \\ \beta_{q-1} & = & \mu_0 + \mu_1 + \cdots + \mu_{q-1} \end{array}$$

and place the sequence

$$\alpha_0, \alpha_1, \ldots, \alpha_{p-1},\ \beta_0, \beta_1, \ldots, \beta_{q-1}$$

in ascending order

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{n-1}.$$

There exist sequences

$$0 = i_0 \leq i_1 \leq \cdots \leq i_n = p$$

and

$$0 = j_0 \leq j_1 \leq \cdots \leq j_n = q$$

such that $i_k + j_k = k$ for $0 \leq k \leq n$ and

$$\gamma_k = (i_{k+1} - i_k)\alpha_{i_k} + (j_{k+1} - j_k)\beta_{j_k}$$

for $0 \leq k \leq n-1$.

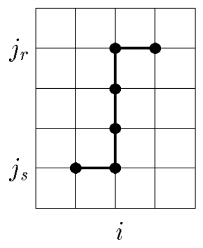
Let us check that

$$x \times y = \sum_{k=0}^{n} (\gamma_k - \gamma_{k-1}) A_{i_k} \times B_{j_k},$$

where $\gamma_{-1} = 0$ and $\gamma_n = 1$. Indeed,

$$\sum_{k=0}^{n} (\gamma_k - \gamma_{k-1}) A_{i_k} = \sum_{i=0}^{p} \sum_{k \in I(i)} (\gamma_k - \gamma_{k-1}) A_i.$$

Let $0 \leq i \leq p$, $s = \min I(i)$, and $r = \max I(i)$.



We see that

$$\begin{array}{lclcl}
\gamma_{s-1} & = & 1\alpha_{i-1} & + & 0\beta_{j_{s-1}} \\
\gamma_s & = & 0\alpha_i & + & 1\beta_{j_s} \\
 & \dots & & & \\
\gamma_{r-1} & = & 0\alpha_i & + & 1\beta_{j_{r-1}} \\
\gamma_r & = & 1\alpha_i & + & 0\beta_{j_r}
\end{array}$$

Hence

$$\sum_{k\in I(i)}(\gamma_k-\gamma_{k-1})=\sum_{k=s}^{r}(\gamma_k-\gamma_{k-1})=\alpha_i-\alpha_{i-1}=\lambda_i.$$

Therefore

$$x=\sum_{k=0}^{n}(\gamma_k-\gamma_{k-1})A_{i_k}.$$

The identity

$$y=\sum_{k=0}^{n}(\gamma_k-\gamma_{k-1})B_{j_k}$$

is proved analogously.

Part 3. Consider two maximal paths

$$i_0\times j_0,\ i_1\times j_1,\ \dots,\ i_n\times j_n$$

and

$$u_0\times v_0,\ u_1\times v_1,\ \dots,\ u_n\times v_n.$$

Let

$$\sum_{k=0}^{n}\lambda_k A_{i_k}\times B_{j_k}=\sum_{k=0}^{n}\mu_k A_{u_k}\times B_{v_k},$$

where

$$\lambda_0\geq 0,\dots,\lambda_n\geq 0,\ \mu_0\geq 0,\dots,\mu_n\geq 0$$

and

$$\lambda_0+\dots+\lambda_n=\mu_0+\dots+\mu_n=1.$$

We claim that

$$\lambda_k=\mu_k$$

in the case $i_k \times j_k = u_k \times v_k$ and

$$\lambda_k = \mu_k = 0$$

in the case $i_k \times j_k \neq u_k \times v_k$.

Note that

$$\sum_{k \in I(i)} \lambda_k = \sum_{k \in U(i)} \mu_k$$

for $0 \leq i \leq p$ and

$$\sum_{k \in J(j)} \lambda_k = \sum_{k \in V(j)} \mu_k$$

for $0 \leq j \leq q$.

We see that

$$i_0 = u_0 = 0$$

$$j_0 = v_0 = 0$$

and

$$i_n = u_n = n$$

$$j_n = v_n = n$$

Let $k_1 \geq 1$ be a minimal number such that

$$i_{k_1} = u_{k_1} = p_1,$$

$$j_{k_1} = v_{k_1} = q_1.$$

If $k_1 = 1$, then

$$I(0) = U(0) = 0$$

or

$$J(0) = V(0) = 0.$$

Hence

$$\lambda_0 = \mu_0.$$

Let $k_1 \geq 2$. We may assume that for $1 \leq k \leq k_1 - 1$ the first path is over the second path. This means than

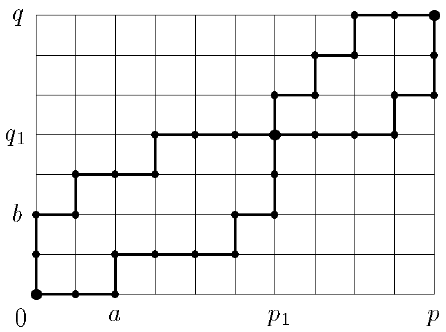
$$i_k < u_k$$

$$j_k > v_k$$

for all $1 \leq k \leq k_1 - 1$. In particular, we have

$$\begin{array}{ll} I(0) = \{0, 1, \ldots, b\} & U(0) = 0, \\ J(0) = 0 & V(0) = \{0, 1, \ldots, a\}, \end{array}$$

where $1 \leq a < p_1$ and $1 \leq b < q_1$.



Thus

$$\begin{aligned}\lambda_0 + \lambda_1 + \cdots + \lambda_b &= \mu_0 \\ \lambda_0 &= \mu_0 + \mu_1 + \cdots + \mu_a\end{aligned}$$

Subtracting the first from the second, we get

$$-(\lambda_1 + \cdots + \lambda_b) = \mu_1 + \cdots + \mu_a.$$

Therefore

$$\lambda_1 = \ldots = \lambda_b = \mu_1 = \ldots = \mu_a = 0$$

and

$$\lambda_0 = \mu_0.$$

Let us check that

$$\lambda_1 = \ldots = \lambda_{k_1-1} = \mu_1 = \ldots = \mu_{k_1-1} = 0.$$

Assume the converse. Let $1 \le m \ge k_1 - 1$ be a minimal number such that $\lambda_m \ne 0$ or $\mu_m \ne 0$. For example, let $\mu_m \ne 0$. Since

$$\mu_1 = \ldots = \mu_a = 0,$$

we have $m \ge a + 1$. We see that

$$j = j_m \ge 1$$

and

$$\max J(j) < \min V(j).$$

Hence

$$1 \le \min J(j) \le \max J(j) \le m - 1.$$

By definition of m, we have

$$\sum_{k \in V(j)} \mu_k = \sum_{k \in J(j)} \lambda_k = 0.$$

Therefore $\mu_m = 0$. The contradiction proves the statement.

Let $k_2 \ge k_1 + 1$ be a minimal number such that

$$i_{k_2} = u_{k_2},$$

$$j_{k_2} = v_{k_2}.$$

Then analogously we obtain

$$\lambda_{k_1} = \mu_{k_1}$$

and

$$\lambda_k = \mu_k = 0$$

for $k_1 + 1 \leq k \leq k_2 - 1$.

We continue the process and obtain m such that $k_m = n$. The last identity

$$\lambda_n = \mu_n$$

follows from the identity

$$\lambda_0 + \cdots + \lambda_n = \mu_0 + \cdots + \mu_n = 1.$$

Part 4. Let

$$P = \{i_0 \times j_0,\ i_1 \times j_1,\ \ldots,\ i_n \times j_n\}$$

be a maximal path, and let

$$S(P) = \operatorname{conv}\{A_{i_0} \times B_{j_0}, A_{i_1} \times B_{j_1}, \ldots, A_{i_n} \times B_{j_n}\}$$

be the maximal simplex corresponding to the maximal path. Denote by T the set consisting of all simplices $S(P)$ with their faces. Then

$$X_p \times Y_q = \bigcup_{\Delta \in T} \Delta.$$

We claim that T is the triangulation of $X_p \times Y_q$.

1^0. Let Γ be a face of Δ and $\Delta \in T$. Obviously $\Gamma \in T$.

2^0. Let $\Gamma, \Delta \in T$ and $\Gamma \subset \Delta$. Let $A_i \times B_j$ be a vertex of Γ, and let $A_{i_0} \times B_{j_0}, \ldots, A_{i_m} \times B_{j_m}$ be all vertices of Δ. Then

$$A_i \times B_j = \sum_{k=0}^{m} \lambda_k A_{i_k} \times B_{j_k},$$

where $\lambda_0 \geq 0, \ldots, \lambda_m \geq 0$ and $\lambda_0 + \cdots + \lambda_m = 1$. From

$$A_i = \sum_{k=0}^{m} \lambda_k A_{i_k}$$

it follows that $\lambda_{k_0} = 1$ for some $0 \leq k_0 \leq m$. This means that $i = i_{k_0}$ and $j = j_{k_0}$. Hence $A_i \times B_j$ is a vertex of Δ. Therefore Γ is a face of Δ.

3^0. Let $\Delta, \Gamma \in T$. Let Δ be the face of the simplex $S(P)$ with

$$P = \{i_0 \times j_0,\ i_1 \times j_1,\ \ldots,\ i_n \times j_n\},$$

and let Γ be the face of the simplex $S(Q)$ with

$$Q = \{u_0 \times v_0,\ u_1 \times v_1,\ \ldots,\ u_n \times v_n\}.$$

Consider a point $x \times y \in \Delta \cap \Gamma$. Then

$$x \times y = \sum_{k=0}^{n} \lambda_k A_{i_k} \times B_{j_k} = \sum_{k=0}^{n} \mu_k A_{u_k} \times B_{v_k},$$

where

$$\lambda_0 \geq 0, \dots, \lambda_n \geq 0, \ \mu_0 \geq 0, \dots, \mu_n \geq 0$$

and

$$\lambda_0 + \dots + \lambda_n = \mu_0 + \dots + \mu_n = 1.$$

By Part 3, we have $\lambda_k = \mu_k$ for all $0 \leq k \leq n$. Moreover, if $\lambda_k \neq 0$, then $i_k \times j_k = u_k \times v_k$. Hence

$$\Delta \cap \Gamma = \mathrm{conv}\{A_i \times B_j : \ A_i \times B_j \in \Delta \cap \Gamma\}$$

This means that $\Delta \cap \Gamma \in T$. •

3. Regular intersections

Let $A_0, A_1, \dots, A_n \in R^n$ be a nonsingular set. Consider a simplex

$$V = \mathrm{conv}\{A_0, A_1, \dots, A_n\}.$$

Let the simplex V be intersected by a plane

$$\Gamma = \{x \in R^n : \ (w, x) = t\},$$

where $w \neq 0$. Put

$$\begin{aligned} V^- &= \{x \in V : \ (w, x) \leq t\}, \\ V^0 &= \{x \in V : \ (w, x) = t\}, \\ V^+ &= \{x \in V : \ (w, x) \geq t\}. \end{aligned}$$

We are going to construct the compatible triangulations

$$\begin{array}{ccc} T^0 & \subset & T^+ \\ \cap & & \cap \\ T^- & \subset & T \end{array}$$

of the polytopes

$$\begin{array}{ccc} V^0 & \subset & V^+ \\ \cap & & \cap \\ V^- & \subset & V \end{array}$$

A level t we call regular if

$$(w, A_i) \neq t$$

for all $0 \leq i \leq n$ and singular if

$$(w, A_i) = t$$

for some $0 \leq i \leq n$. In the section we consider regular intersections.

Let t be regular. We assume that

$$(w, A_i) \leq (w, A_j)$$

for all $0 \leq i \leq j \leq n$. Then

$$(w, A_{i-1}) < t$$

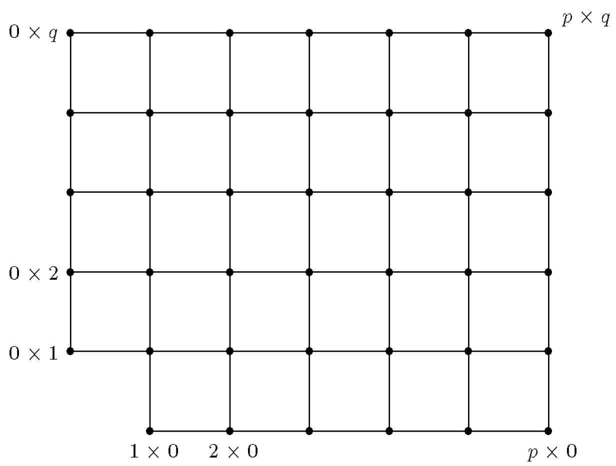
for $i = 1, \ldots, p$ and

$$(w, A_{n-j+1}) < t$$

for $j = 1, \ldots, q$, where $p + q = n + 1$. Define

$$\begin{array}{rcll} A_{i0} & = & A_{i-1} & \text{for} \quad 1 \leq i \leq p \\ A_{0j} & = & A_{n-j-1} & \text{for} \quad 1 \leq j \leq q \end{array}$$

Let G_{pq} be the graph



In the nontrivial case $p \geq 1$ and $q \geq 1$ the graph G_{pq} has two initial vertices 1×0 and 0×1, and one final vertex $p \times q$. Hence the graph has $C^p_{p+q-1} + C^q_{p+q-1}$ maximal paths.

Let $G^-_{pq} \subset G_{pq}$ be the graph with vertices

$$i \times j : \; 1 \leq i \leq p \; \text{ and } \; 0 \leq j \leq q,$$

let $G^0_{pq} \subset G_{pq}$ be the graph with vertices

$$i \times j : \; 1 \leq i \leq p \; \text{ and } \; 1 \leq j \leq q,$$

and let $G^+_{pq} \subset G_{pq}$ be the graph with vertices

$$i \times j : \; 0 \leq i \leq p \; \text{ and } \; 1 \leq j \leq q.$$

Denote by A_{ij}, where $1 \le i \le p$ and $1 \le j \le q$, the point of intersection of the interval $[A_{i0}, A_{0j}]$ with the plane Γ. We see that

$$A_{ij} = (1-\lambda)A_{i0} + \lambda A_{0j},$$

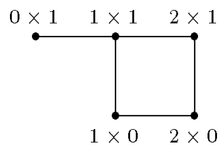
where

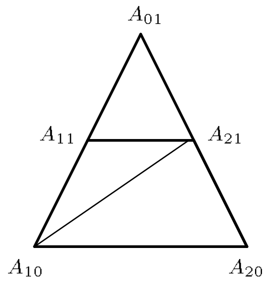
$$\lambda = \frac{t-(w, A_{i0})}{(w, A_{0j}) - (w, A_{i0})}.$$

Let the vertex $i \times j \in G_{pq}$ be associated with the point $A_{ij} \in V$.

We claim that the triangulations T, T^-, T^0, T^+ associated with the graphs G_{pq}, G^-_{pq}, G^0_{pq}, G^+_{pq} are the compatible triangulations of the polytopes V, V^-, V^0, V^+.

Examples. For $p = 2$ and $q = 1$ we have

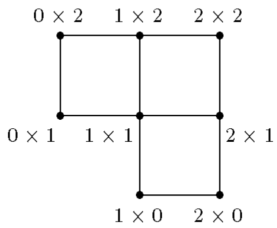


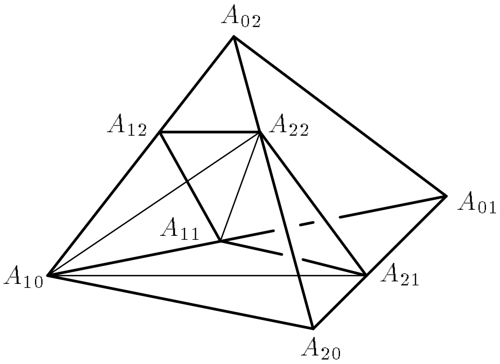


Note the simplicial homeomorphisms

$$\begin{array}{lcl} V^- & \approx & \Delta^1 \times \Delta^1 \\ V^0 & \approx & \Delta^1 \times \Delta^0 \\ V^+ & \approx & \Delta^2 \times \Delta^0 \end{array}$$

For $p = 2$ and $q = 2$ we have





Note the simplicial homeomorphisms

$$\begin{array}{lcl} V^- & \approx & \Delta^1 \times \Delta^2 \\ V^0 & \approx & \Delta^1 \times \Delta^1 \\ V^+ & \approx & \Delta^2 \times \Delta^1 \end{array}$$

Let

$$\begin{array}{rcl} E_1^n &=& (1,0,0,\dots,0) \\ E_2^n &=& (0,1,0,\dots,0) \\ &\cdots& \\ E_n^n &=& (0,0,0,\dots,1) \end{array}$$

be unit vectors of $\mathbb{R}^n$.

Theorem 2 *Let t be regular. The triangulations T, T^-, T^0, T^+ associated with the graphs G_{pq}, G_{pq}^-, G_{pq}^0, G_{pq}^+ are the compatible triangulations of the polytopes V, V^-, V^0, V^+. The mappings*

$$\begin{array}{lllll} F_- : & E_i^p \times E_{j+1}^{q+1} & \longrightarrow & A_{ij} & \textit{for} \quad 1 \le i \le p,\ 0 \le j \le q \\ F_0 : & E_i^p \times E_j^q & \longrightarrow & A_{ij} & \textit{for} \quad 1 \le i \le p,\ 1 \le j \le q \\ F_+ : & E_{i+1}^{p+1} \times E_j^q & \longrightarrow & A_{ij} & \textit{for} \quad 0 \le i \le p,\ 1 \le j \le q \end{array}$$

are extended to the simplicial homeomorphisms

$$\begin{array}{llll} F_- : & \Delta^{p-1} \times \Delta^q & \longrightarrow & V^- \\ F_0 : & \Delta^{p-1} \times \Delta^{q-1} & \longrightarrow & V^0 \\ F_+ : & \Delta^p \times \Delta^{q-1} & \longrightarrow & V^+ \end{array}$$

Proof. Let us prove the statement for the polytope V^0. Without loss of generality it can be assumed that $t = 0$. Let

$$\begin{array}{rcl} t_i &=& -(w, A_{i0}), \\ \tau_j &=& (w, A_{0j}). \end{array}$$

Then $t_i > 0$ and $\tau_j > 0$. Obviously

$$A_{ij} = \frac{\tau_j}{t_i + \tau_j} A_{i0} + \frac{t_i}{t_i + \tau_j} A_{0j}.$$

Consider a maximal path

$$i_1 \times j_1,\ \dots,\ i_n \times j_n$$

on the graph G_{pq}^0. We note that

$$1 = i_1 \le \dots \le i_n = p$$

and

$$1 = j_1 \le \dots \le j_n = q,$$

where $i_k + j_k = k + 1$ for all $1 \le k \le n$. Let

$$\begin{array}{rcl} I(i) &=& \{k : \ i_k = i\}, \\ J(j) &=& \{k : \ j_k = j\}, \end{array}$$

where $1 \le i \le p$ and $1 \le j \le q$.

We claim that the set $A_{i_1 j_1}, \dots, A_{i_n j_n}$ is nonsingular. Assume that

$$\lambda_1 A_{i_1 j_1} + \dots + \lambda_n A_{i_n j_n} = 0$$

and

$$\lambda_1 + \dots + \lambda_n = 0.$$

Since the set $A_0, A_1, \dots, A_n$ is nonsingular, we have

$$\sum_{k \in I(i)} \frac{\tau_{j_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = 0$$

for $1 \le i \le p$ and

$$\sum_{k \in J(j)} \frac{t_{i_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = 0$$

for $1 \le j \le q$. From the system of linear equations we obtain

$$\lambda_1 = \dots = \lambda_n = 0.$$

This means that the set $A_{i_1 j_1}, \dots, A_{i_n j_n}$ is nonsingular.

Let $x \in V^0$. There exists a maximal path

$$i_1 \times j_1, \ \dots, \ i_n \times j_n$$

such that

$$x \in \mathrm{conv}\{A_{i_1 j_1}, \dots, A_{i_n j_n}\}.$$

Indeed, let

$$x = \zeta_1 A_{10} + \dots + \zeta_p A_{p0} + \eta_1 A_{01} + \dots + \eta_q A_{0q}.$$

Since $(w, x) = 0$, we have

$$\zeta_1 t_1 + \dots + \zeta_p t_p = \eta_1 \tau_1 + \dots + \eta_q \tau_q = r$$

for some $r > 0$. Let

$$u_i = \frac{1}{r} t_i \zeta_i$$

for $1 \le i \le p$ and

$$v_j = \frac{1}{r} \tau_j \eta_j$$

for $1 \le j \le q$. Then

$$u \times v \in \Delta^{p-1} \times \Delta^{q-1}.$$

By Theorem 1, there exists a maximal path

$$i_1 \times j_1, \ \dots, \ i_n \times j_n$$

such that

$$u \times v \in \operatorname{conv}(E^p_{i_1} \times E^q_{j_1}, \ldots, E^p_{i_n} \times E^q_{j_n}).$$

Then

$$u \times v = \sum_{k=1}^{n} \gamma_k E^p_{i_k} \times E^q_{j_k}$$

for $\gamma_1 \geq 0, \ldots, \gamma_n \geq 0$ and $\gamma_1 + \cdots + \gamma_n = 1$. Therefore,

$$x = \sum_{k=1}^{n} \lambda_k A_{i_k j_k},$$

where

$$\lambda_k = r\frac{t_{i_k} + \tau_{j_k}}{t_{i_k}\tau_{j_k}}\gamma_k.$$

Indeed,

$$\begin{aligned}
\sum_{k=1}^{n} \lambda_k A_{i_k j_k} &= \sum_{k=1}^{n} r\frac{t_{i_k} + \tau_{j_k}}{t_{i_k}\tau_{j_k}}\gamma_k \left(\frac{\tau_{j_k}}{t_{i_k} + \tau_{j_k}} A_{i_k 0} + \frac{t_{i_k}}{t_{i_k} + \tau_{j_k}} A_{0 j_k}\right) \\
&= \sum_{k=1}^{n} \frac{r\gamma_k}{t_{i_k}} A_{i_k 0} + \sum_{k=1}^{n} \frac{r\gamma_k}{\tau_{j_k}} A_{0 j_k} \\
&= \sum_{i=1}^{p} \frac{r}{t_i} \left(\sum_{k \in I(i)} \gamma_k\right) A_{i0} + \sum_{j=1}^{q} \frac{r}{\tau_i} \left(\sum_{k \in J(j)} \gamma_k\right) A_{0j} \\
&= \sum_{i=1}^{p} \frac{r}{t_i} u_i A_{i0} + \sum_{j=1}^{q} \frac{r}{\tau_i} v_j A_{0j} \\
&= \sum_{i=1}^{p} \zeta_i A_{i0} + \sum_{j=1}^{q} \eta_i A_{0j} \ = \ x
\end{aligned}$$

Analogously

$$\begin{aligned}
\sum_{k=1}^{n} \lambda_k &= \sum_{k=1}^{n} r\frac{t_{i_k} + \tau_{j_k}}{t_{i_k}\tau_{j_k}}\gamma_k \\
&= \sum_{k=1}^{n} \frac{r\gamma_k}{t_{i_k}} + \sum_{k=1}^{n} \frac{r\gamma_k}{\tau_{j_k}} \\
&= \sum_{i=1}^{p} \frac{r}{t_i} \left(\sum_{k \in I(i)} \gamma_k\right) + \sum_{j=1}^{q} \frac{r}{\tau_i} \left(\sum_{k \in J(j)} \gamma_k\right) \\
&= \sum_{i=1}^{p} \frac{r}{t_i} u_i + \sum_{j=1}^{q} \frac{r}{\tau_i} v_j \\
&= \sum_{i=1}^{p} \zeta_i + \sum_{j=1}^{q} \eta_i \ = \ 1
\end{aligned}$$

Consider two maximal paths

$$i_1 \times j_1, \ \ldots, \ i_n \times j_n$$

and

$$u_1 \times v_1,\ \ldots,\ u_n \times v_n.$$

Let

$$\sum_{k=1}^{n} \lambda_k A_{i_k j_k} = \sum_{k=1}^{n} \mu_k A_{u_k v_k}$$

where

$$\lambda_1 \geq 0, \ldots, \lambda_n \geq 0,\ \mu_1 \geq 0, \ldots, \mu_n \geq 0$$

and

$$\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n = 1.$$

We claim that

$$\lambda_k = \mu_k$$

in the case $i_k \times j_k = u_k \times v_k$ and

$$\lambda_k = \mu_k = 0$$

in the case $i_k \times j_k \neq u_k \times v_k$.

Indeed, we see that

$$\sum_{k \in I(i)} \frac{\tau_{j_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = \sum_{k \in U(i)} \frac{\tau_{v_k}}{t_{u_k} + \tau_{v_k}} \mu_k$$

for $1 \leq i \leq p$ and

$$\sum_{k \in J(j)} \frac{t_{i_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = \sum_{k \in V(j)} \frac{t_{u_k}}{t_{u_k} + \tau_{v_k}} \mu_k$$

при $1 \leq j \leq q$. Then we may continue as in Part 3 of the proof of Theorem 1.

Let

$$P = \{i_1 \times j_1,\ \ldots,\ i_n \times j_n\}$$

be a maximal path, and let

$$S(P) = \operatorname{conv}\{A_{i_1 j_1}, \ldots, A_{i_n j_n}\}$$

be the simplex corresponding to the path. Denote by T^0 the set consisting of all simplices $S(P)$ with their faces. Then

$$V^0 = \bigcup_{\Delta \in T} \Delta.$$

Obviously T^0 is the triangulation of V^0. Besides, the mapping

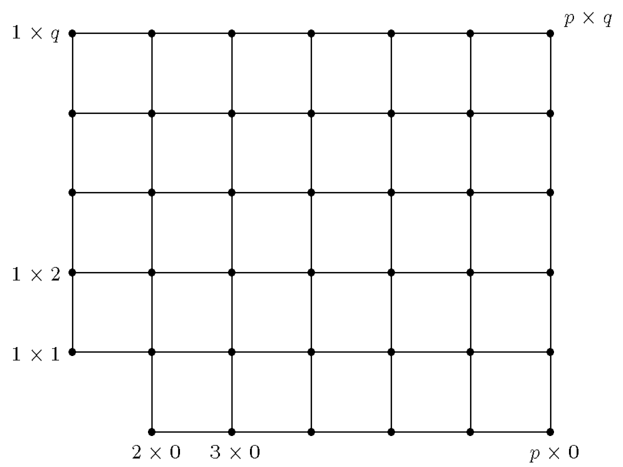
$$F_0 :\ \Delta^{p-1} \times \Delta^{q-1} \longrightarrow V^0$$

is the simplicial homeomorphism.

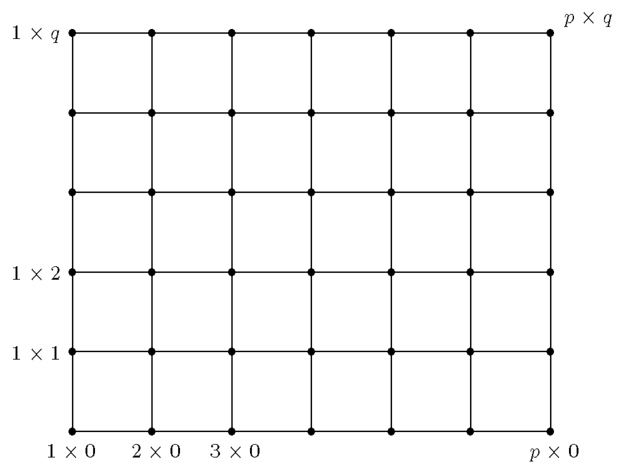
Let us prove the statement about the compatible triangulations by induction on n. For $n = 1$ the statement is obvious. Assume that for $n - 1$ the is statement true. By the induction hypothesis, the statement is true for the simplex

$$U = \operatorname{conv}\{A_1, \ldots, A_n\}.$$

Therefore the triangulation $\tilde{T}$ associated with the graph



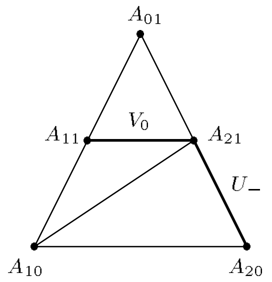
is a triangulation of the polytope $V_0 \cup U_-$. Hence the triangulation T^- associated with the graph G^-_{pq} of the form



is a tiangulation of the polytope V^-. The triangulation T^- consists of all simplices

$$\Delta = \mathrm{conv}\{A_{10}, \Gamma\},$$

where $\Gamma \in \tilde{T}$, and their faces. The construction of the triangulation T^- is shown in the picture



We analogously prove that the triangulation T^+ associated with the graph G^+_{pq} is a tiangulation of the polytope V^+. $\bullet$

4. Singular intersections

We consider the same simplex

$$V = \mathrm{conv}\{A_0, A_1, \ldots, A_n\}$$

as in the previous section. In this section we denote the plane in the form

$$\Gamma_t = \{x \in R^n : \ (w, x) = t\}$$

and the polytopes in the form

$$\begin{array}{rcl} V_t^- & = & \{x \in V : \ (w, x) \leq t\}, \\ V_t^0 & = & \{x \in V : \ (w, x) = t\}, \\ V_t^+ & = & \{x \in V : \ (w, x) \geq t\}. \end{array}$$

Let t^0 be a fixed singular level. We assume that

$$(w, A_i) \leq (w, A_j)$$

for all $0 \leq i \leq j \leq n$. Then

$$(w, A_{i-1}) < t^0$$

for $i = 1, \ldots, p$,

$$(w, A_{n-j+1}) > t^0$$

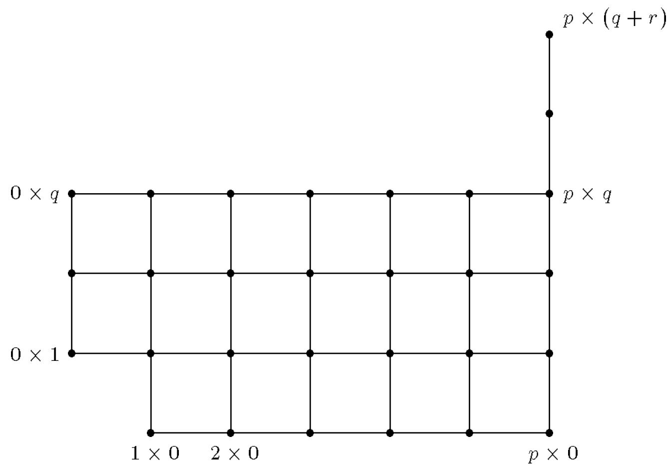
for $j = 1, \ldots, q$, and

$$(w, A_{n-q-k}) = t^0$$

for $k = 1, \ldots, r$, where $p + q + r = n + 1$. Define

$$\begin{array}{rcll} A_{i0} & = & A_{i-1} & \text{for} \quad 1 \leq i \leq p \\ A_{0j} & = & A_{n-j-1} & \text{for} \quad 1 \leq j \leq q \\ A_{pk} & = & A_{n-q-k} & \text{for} \quad 1 \leq k \leq r \end{array}$$

Let G_{pqr} be the graph



To obtain the graph G_{pqr} we add vertices

$$p \times (q+1), p \times (q+2), \dots, p \times (q+r)$$

and corresponding vertical edges to the graph G_{pq}. Analogously we define the graphs G^-_{pqr}, G^0_{pqr} and G^+_{pqr}.

Denote by A_{ij}, where $1 \le i \le p$ and $1 \le j \le q$, the point of intersection of the interval $[A_{i0}, A_{0j}]$ with the plane Γ_{t^0}. Let the vertex $i \times j \in G_{pqr}$ be associated with the point $A_{ij} \in V$.

Theorem 3 *Let t^0 be singular. The triangulations associated with graphs $G_{pqr}, G^-_{pqr}, G^0_{pqr}, G^+_{pqr}$ are the compatible triangulations of the polytopes V, $V^-_{t^0}$, $V^0_{t^0}$, $V^+_{t^0}$. The mappings*

$$F_- : \begin{cases} E^p_i \times E^{q+1}_{j+1} \times 0^r \longrightarrow A_{ij} & for \quad 1 \le i \le p,\ 0 \le j \le q \\ 0^p \times 0^{q+1} \times E^r_k \longrightarrow A_{p,q+k} & for \quad 1 \le k \le r \end{cases}$$

$$F_0 : \begin{cases} E^p_i \times E^q_j \times 0^r \longrightarrow A_{ij} & for \quad 1 \le i \le p,\ 1 \le j \le q \\ 0^p \times 0^q \times E^r_k \longrightarrow A_{p,q+k} & for \quad 1 \le k \le r \end{cases}$$

$$F_+ : \begin{cases} E^{p+1}_{i+1} \times E^q_j \times 0^r \longrightarrow A_{ij} & for \quad 0 \le i \le p,\ 1 \le j \le q \\ 0^{p+1} \times 0^q \times E^r_k \longrightarrow A_{p,q+k} & for \quad 1 \le k \le r \end{cases}$$

are extanded to the simplicial homeomorphisms

$$\begin{array}{lll} F_- : & (\Delta^{p-1} \times \Delta^q) * \Delta^{r-1} & \longrightarrow V^-_{t^0} \\ F_0 : & (\Delta^{p-1} \times \Delta^{q-1}) * \Delta^{r-1} & \longrightarrow V^0_{t^0} \\ F_+ : & (\Delta^p \times \Delta^{q-1}) * \Delta^{r-1} & \longrightarrow V^+_{t^0} \end{array}$$

Proof. Let us prove the statement for the polytope $V^0_{t^0}$. Without loss of generality it can be assumed that $t^0 = 0$. Let

$$\begin{array}{rcl} t_i & = & -(w, A_{i0}), \\ \tau_j & = & (w, A_{0j}), \end{array}$$

where $1 \le i \le p$ and $1 \le j \le q$. Then $t_i > 0$ and $\tau_j > 0$. Obviously

$$A_{ij} = \frac{\tau_j}{t_i + \tau_j} A_{i0} + \frac{t_i}{t_i + \tau_j} A_{0j}$$

where $1 \le i \le p$ and $1 \le j \le q$.

Consider a maximal path

$$i_1 \times j_1,\ \dots,\ i_n \times j_n$$

on the graph G^0_{pqr}. Let $m = p + q$. We note that

$$1 = i_1 \le \dots \le i_m = p$$

and

$$1 = j_1 \leq \cdots \leq j_m = q,$$

where $i_k + j_k = k + 1$ for all $1 \leq k \leq m$. Let

$$\begin{array}{rcl} I(i) & = & \{k : \ i_k = i\}, \\ J(j) & = & \{k : \ j_k = j\}, \end{array}$$

where $1 \leq i \leq p$ and $1 \leq j \leq q$.

We claim that the set $A_{i_1 j_1}, \ldots, A_{i_n j_n}$ is nonsingular. Assume that

$$\lambda_1 A_{i_1 j_1} + \cdots + \lambda_n A_{i_n j_n} = 0$$

and

$$\lambda_1 + \cdots + \lambda_n = 0.$$

Since the set $A_0, A_1, \ldots, A_n$ is nonsingular, we have

$$\sum_{k \in I(i)} \frac{\tau_{j_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = 0$$

for $1 \leq i \leq p$,

$$\sum_{k \in J(j)} \frac{t_{i_k}}{t_{i_k} + \tau_{j_k}} \lambda_k = 0$$

for $1 \leq j \leq q$, and

$$\lambda_k = 0$$

for $m + 1 \leq k \leq n$. From the system of linear equations we obtain

$$\lambda_1 = \ldots = \lambda_n = 0.$$

This means that the set $A_{i_1 j_1}, \ldots, A_{i_n j_n}$ is nonsingular.

Let $x \in V^0_{t^0}$. There exists a maximal path

$$i_1 \times j_1, \ \ldots, \ i_n \times j_n$$

such that

$$x \in \operatorname{conv}\{A_{i_1 j_1}, \ldots, A_{i_n j_n}\}.$$

Indeed, let

$$x = \zeta_1 A_{10} + \cdots + \zeta_p A_{p0} + \eta_1 A_{01} + \cdots + \eta_q A_{0q} + \theta_1 A_{p1} + \cdots + \theta_r A_{pr}.$$

Since

$$(w, x) = (w, A_{p1}) = \ldots = (w, A_{pr}) = 0$$

we have

$$\zeta_1 t_1 + \cdots + \zeta_p t_p = \eta_1 \tau_1 + \cdots + \eta_q \tau_q = r$$

for some $r > 0$. Then we continue as in the proof of Theorem 2.

Consider two maximal paths

$$i_1 \times j_1, \ \ldots, \ i_n \times j_n$$

and

$$u_1 \times v_1, \ \ldots, \ u_n \times v_n.$$

Let

$$\sum_{k=1}^{n} \lambda_k A_{i_k j_k} = \sum_{k=1}^{n} \mu_k A_{u_k v_k}$$

where

$$\lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \ \mu_1 \geq 0, \ldots, \mu_n \geq 0$$

and

$$\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n = 1.$$

We see that

$$\lambda_k = \mu_k$$

in the case $i_k \times j_k = u_k \times v_k$ and

$$\lambda_k = \mu_k = 0$$

in the case $i_k \times j_k \neq u_k \times v_k$.

Let

$$P = \{i_1 \times j_1, \ \ldots, \ i_n \times j_n\}$$

be a maximal path, and let

$$S(P) = \mathrm{conv}\{A_{i_1 j_1}, \ldots, A_{i_n j_n}\}$$

be the simplex corresponding to the path. Denote by T^0 the set consisting of all simplices $S(P)$ with their faces. Then

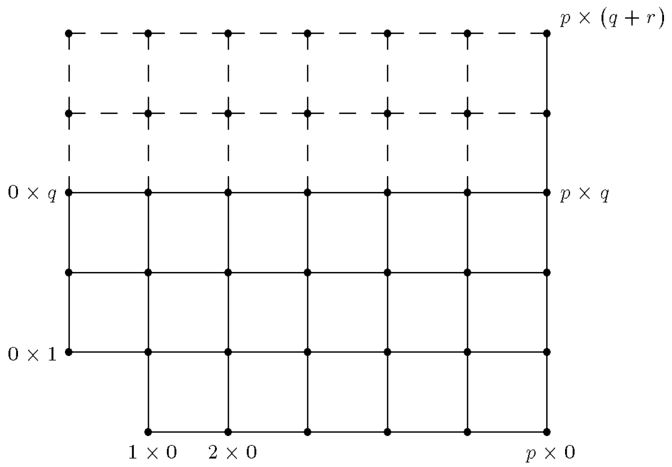
$$V^0_{t^0} = \bigcup_{\Delta \in T} \Delta.$$

Obviously T^0 is the triangulation of $V^0_{t^0}$. Besides, the mapping

$$F_0 : \ (\Delta^{p-1} \times \Delta^{q-1}) * \Delta^{r-1} \longrightarrow V^0_{t^0}$$

is the simplicial homeomorphism. •

Remark. The last theorem may be proved by continuity. For example, consider the polytope $V^0_{t_0}$. By continuity, there exists $\varepsilon > 0$ such that t is regular whenever $t_0 - \varepsilon < t < t_0$. By Theorem 2, the polytope V^0_t has a triangulation associated with the graph $G_{p,q+r}$.



We note that $G^0_{p,q,r}$ is a subgraph of $G^0_{p,q+r}$. Obviously

$$\lim_{t \to t_0} A^t_{ij} = A_{ij}$$

for all $1 \le i \le p$ and $1 \le j \le q + r$. Consider a maximal path $i_1 \times j_1, \ldots, i_n \times j_n$ on the graph $G^0_{p,q+r}$. By Theorem 2, the set

$$A^t_{i_1,j_1}, \ldots, A^t_{i_n,j_n}$$

is nonsingular. Obviously

$$\lim_{t \to t_0}(A^t_{i_1,j_1}, \ldots, A^t_{i_n,j_n}) = (A_{i_1,j_1}, \ldots, A_{i_n,j_n}).$$

Let m be a maximal number such that $j_k = q$. The set

$$A_{i_1,j_1}, \ldots, A_{i_m,j_m}, A_{p,q+1}, \ldots, A_{p,q+r}$$

is nonsingular and

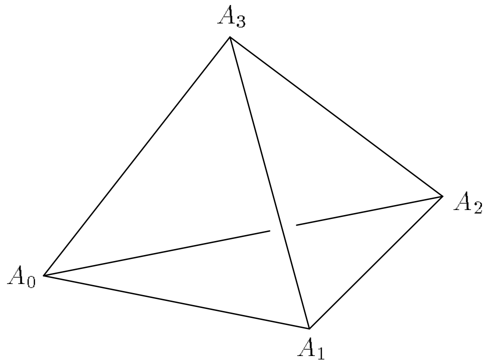
$$\mathrm{conv}\{A_{i_1,j_1}, \ldots, A_{i_n,j_n}\} = \mathrm{conv}\{A_{i_1,j_1}, \ldots, A_{i_m,j_m}, A_{p,q+1}, \ldots, A_{p,q+r}\}.$$

We see that the right simplex belongs to the triangulation of the polytope $V^0_{t^0}$.

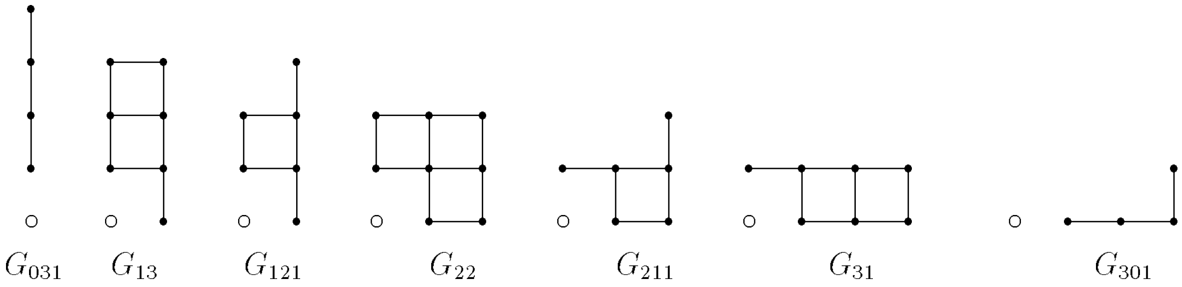
Example. Let

$$V = \mathrm{conv}\{A_0, A_1, A_2, A_3\}$$

be a three-dimensional simplex



and let $t_0 < t_1 < t_2 < t_3$ be singular levels. When the level t increases from t_0 to t_3, we see the following graphs $G_{p,q}$ and $G_{p,q,r}$.



Here $\circ$ is the origin.

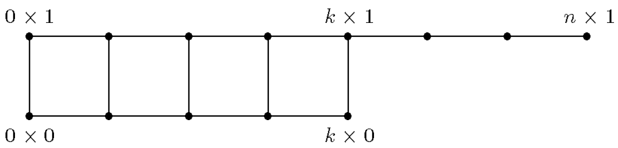
5. Cylinders of simplicial mappings

Let V be a simplex of dimension n, let S be a simplex of dimension k, and let

$$F: V \longrightarrow S$$

be a simplicial surjection. Recall the notion of the cylinder Cyl$F$ of the mapping F. Let the equivalence relation $\sim$ in the cylinder cyl$V = V \times [0,1]$ have the following classes of equivalence $[x \times t]$. If $0 < t \leq 1$, then $[x \times t] = x \times t$. If $t = 0$, then $[x \times 0] = F^{-1}\{F(x)\} \times 0$. By definition, the cylinder Cyl$F$ of the mapping F is the factor-space cyl$V/\sim$.

We are going to prove that the cylinder Cyl$F$ is homeomorphic to the ball D^{n+1}. Namely, we prove that the cylinder Cyl$F$ is homeomorphic to the join cyl$\Delta^k * \Delta^{n-k-1}$. In particular, the cylinder Cyl$F$ has a natural triangulation associated with the graph



The triangulation has $k+1$ simplices of dimension $n+1$.

In the cases $k = n$ and $k = 0$ the statement is obvious. In the case $k = n$ the mapping F is the simplicial homeomorphism. Hence

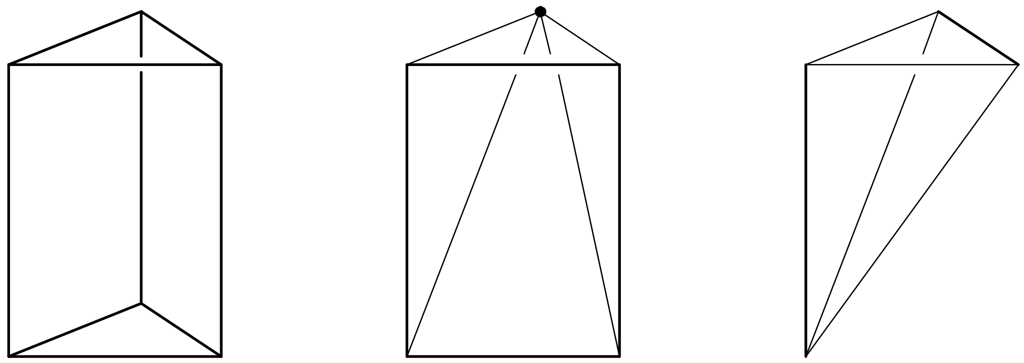
$$\mathrm{Cyl}F \approx \Delta^n \times [0,1].$$

Therefore Cyl$F$ has a triangulation with $n+1$ simplicies of dimension $n+1$. In the case $k = 0$ the image of the mapping F is a point. Hence

$$\mathrm{Cyl}F \approx \Delta^{n+1}.$$

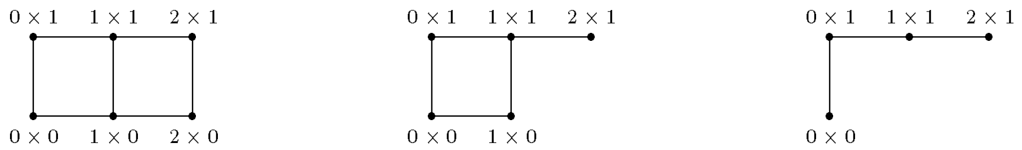
Therefore Cyl$F$ has a triangulation with 1 simplex of dimension $n+1$.

Example. Let $n = 2$. The cases $k = 2, 1, 0$ are shown in the pictures



cyl $\Delta^2 * \emptyset$ cyl $\Delta^1 * \Delta^0$ cyl $\Delta^0 * \Delta^1$

In the first picture we see the cylinder cylΔ^2. In the second picture we see the join of the cylinder cylΔ^1 and the simplex Δ^0. In the third picture we see the join of the cylinder cylΔ^0 and the simplex Δ^1. The triangulations are associated with the graphs



Theorem 4 *The cylinder* Cyl$F$ *of the simplicial surjection*

$$F : V \longrightarrow S$$

is homeomorphic to the join cyl$\Delta^k * \Delta^{n-k-1}$.

Proof. We construct a convex $(n+1)$-dimensional polytope M and a continuous surjection

$$\Phi : \text{ cyl}V \longrightarrow M$$

such that

$$\text{Cyl}F = \text{cyl}V/\text{zer}\Phi.$$

Here zerΦ is the equivalence relation in the cylinder cyl$V$ such that $x \times t$ is equivalent to $x' \times t'$ if and only if $\Phi(x \times t) = \Phi(x' \times t')$. Then the proof of the theorem follows from the homeomorphism

$$\text{cyl}V/\text{zer}\Phi \approx M.$$

The representation

$$M = \text{cyl}\Delta^k * \Delta^{n-k-1}$$

is obtained automatically.

We may assume that $S \subset V \subset \mathbb{R}^n$. Let $p = k+1$, and let $A_{10}, \ldots, A_{p0}$ be the vertices of the simplex S. Let $p + q = n+1$, and let $A_{10}, \ldots, A_{p0}, A_{01}, \ldots, A_{0q}$ be the vertices of the simplex V. We may assume that

$$F(A_{i0}) = A_{i0}$$

for $1 \le i \le p$ and

$$F(A_{0j}) = A_{\nu(j)0}$$

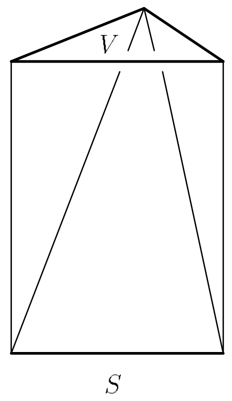
for $1 \le j \le q$, where

$$\nu: \ \{1, 2, \ldots, q\} \longrightarrow \{1, 2, \ldots, p\}.$$

Let us consider the polytope

$$M = \mathrm{conv}\{A_{10} \times 0, \ldots, A_{p0} \times 0; \ A_{10} \times 1, \ldots, A_{p0} \times 1; \ A_{01} \times 1, \ldots, A_{0q} \times 1\}.$$

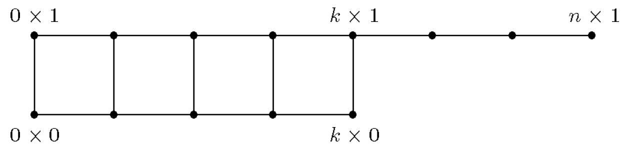
The polytope M is shown in the pucture



The polytope M has a triangulation with $k+1$ simplices of dimension $n+1$. The simplices have the form

$$\mathrm{conv}\{A_{10} \times 0, \ldots, A_{i0} \times 0, A_{i0} \times 1, \ldots, A_{p0} \times 1, A_{01} \times 1, \ldots, A_{0q} \times 1\},$$

where $1 \le i \le p$. This means that the triangulation is associated with the graph.



In particular, the triangulation of the polytope M is the same as the triangulation of the join $\mathrm{cyl}\Delta^k * \Delta^{n-k-1}$.

Let $w \in \mathbb{R}^n$ be any vector such that

$$(w, A_{i0} - A_{10}) = 0$$

for $1 \le i \le p$,

$$(w, A_{0j} - A_{01}) = 0$$

for $1 \le j \le q$, and

$$(w, A_{10}) < (w, A_{01}).$$

Consider the plane

$$\Gamma_t = \{x \in R^n : \ (w, x) = (1-t)(w, A_{10}) + t(w, A_{01})\}$$

and the polytopes

$$\begin{aligned} V_t^- &= \{x \in V : \ (w, x) \le (1-t)(w, A_{10}) + t(w, A_{01})\}, \\ V_t^+ &= \{x \in V : \ (w, x) \ge (1-t)(w, A_{10}) + t(w, A_{01})\}. \end{aligned}$$

We see that $V_0^- = S$ and $V_1^- = V$

Denote by A_{ij}^t, where $1 < t < 1$, the point of intersection of the interval $[A_{i0}, A_{0j}]$ with the plane Γ_t. It is clear that

$$A_{ij}^t = (1-t)A_{i0} + tA_{0j}.$$

Let

$$B_{ij}^t = (1-t^2)A_{i0} + t^2 A_{0j}.$$

We note that

$$\frac{\|B_{ij}^t - A_{i0}\|}{\|A_{ij}^t - A_{i0}\|} = \frac{\|A_{ij}^t - A_{i0}\|}{\|A_{0j} - A_{i0}\|}.$$

Using the compatible triangulations of the polytopes V_t^-, V_t^+, V, define the simplicial mapping

$$\varphi_t : \ V \longrightarrow V_t^-$$

by the rule

$$\begin{array}{lll} A_{i0} & \longmapsto & A_{i0} \\ A_{0j} & \longmapsto & A_{\nu(j)j}^t \\ A_{ij}^t & \longmapsto & A_{ij}^t \quad \text{for} \quad i \ne \nu(j) \\ A_{ij}^t & \longmapsto & B_{ij}^t \quad \text{for} \quad i = \nu(j) \end{array}$$

If $0 < t < 1$, then φ_t is a homeomorphism. Let $\varphi_1 = \mathrm{id}$ and $\varphi_0 = F$, where id is the identical mapping.

Example. Let $n = 2$ and $k = 1$. Let the mapping

$$F : \ V \longrightarrow S$$

be such that

$$\begin{array}{lll} A_{10} & \longmapsto & A_{10}, \\ A_{20} & \longmapsto & A_{20}, \\ A_{01} & \longmapsto & A_{10}. \end{array}$$

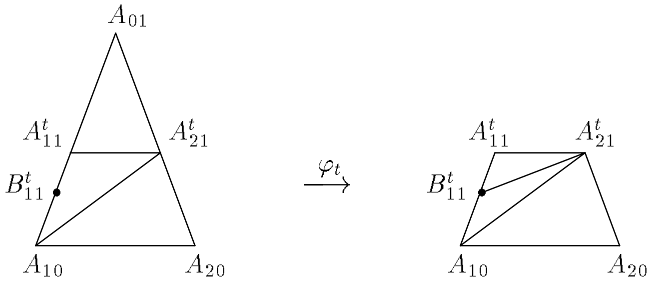
Then

$$\nu : \ 1 \longmapsto 1$$

and the homeomorphism φ_t is such that

$$\begin{array}{ccc} A_{10} & \longmapsto & A_{10}, \\ A_{20} & \longmapsto & A_{20}, \\ A_{01} & \longmapsto & A^t_{11}, \\ A^t_{21} & \longmapsto & A^t_{21}, \\ A^t_{11} & \longmapsto & B^t_{11}. \end{array}$$

The homeomorphism φ_t is shown in the picture



We see that $x \times t \in M$ if and only if $x \in V_t^-$ and $0 \le t \le 1$. Define the surjection

$$\Phi : \ \mathrm{cyl} V \longrightarrow M$$

by the rule

$$\Phi : \ x \times t \longmapsto \varphi_t(x) \times t.$$

We claim that Φ is continuous. To prove this we use the following statement. Let X, Y, Z be topologycal spaces and let

$$\begin{array}{l} f : \ X \longrightarrow Y \\ h : \ Y \longrightarrow Z \end{array}$$

be surjections. Let X be compact and let Y be Hausdorff. If f and $g = h \circ f$ are continuous, then h is continuous.

Let

$$P = \{i_0 \times j_0, \ i_1 \times j_1, \ \ldots, \ i_n \times j_n\}$$

be a maximal path on the graph G_{pq}. Define the continuous mapping

$$f_P : \ \mathrm{cyl}\Delta^n \longrightarrow \mathrm{cyl} V$$

by the rule

$$f_P : \ \lambda \times t \longmapsto \mu \times t,$$

where

$$\mu = \lambda_0 A^t_{i_0 j_0} + \cdots + \lambda_n A^t_{i_n j_n}.$$

Let

$$C^t_{ij} = \begin{cases} A^t_{ij} & \text{for} \quad i \ne \nu(j) \\ B^t_{ij} & \text{for} \quad i = \nu(j) \end{cases}$$

Define the continuous mapping

$$g_P : \mathrm{cyl}\Delta^n \longrightarrow M$$

by the rule

$$g_P : \lambda \times t \longmapsto \mu \times t,$$

where

$$\mu = \lambda_0 C^t_{i_0 j_0} + \cdots + \lambda_n C^t_{i_n j_n}.$$

We see that

$$g_P = \Phi \circ f_P.$$

By the above statement, the mapping

$$\Phi|_{f_P\{\mathrm{cyl}\Delta^n\}} : f_P\{\mathrm{cyl}\Delta^n\} \longrightarrow M.$$

is continuous. Since

$$\mathrm{cyl}V = \bigcup_P f_P\{\mathrm{cyl}\Delta^n\},$$

the mapping Φ is continuous.

Let

$$\mathrm{fact}\Phi : \mathrm{cyl}V/\mathrm{zer}\Phi \longrightarrow M$$

be the mapping such that

$$\mathrm{fact}\Phi : [x \times t] \longmapsto \Phi(x \times t).$$

Since cyl$V$ is compact and since M is Hausdorff, the mapping factΦ is a homeomorphism. We see that

$$\mathrm{cyl}V/\mathrm{zer}\Phi = \mathrm{Cyl}F.$$

Hence Cyl$F \approx M$. •

Bibliography

[1] V.A.Rokhlin, D.B.Fuchs. An innitial course in topology. Moscow, "Nauka", 1977.