Weierstrass functions $\wp_3, \wp_5, \ldots$

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The hyperelliptic curve

$$y^2 = (2n)!^2 x^{2n+1} - g_2 x^{2n-1} - g_3 x^{2n-2} - \cdots - g_{2n+1}$$

has the uniformization $x = \wp_{2n+1}(u), \;\; y = \wp_{2n+1}^{(2n-1)}(u)$.

Contents

Introduction

The elliptic curve

$$y^2 = 4x^3 - g_2 x - g_3$$

has the uniformization

$$\begin{aligned} x &= \wp(u), \\ y &= \wp'(u), \end{aligned}$$

where

$$\wp(u) = u^{-2} + c_2 u^2 + c_4 u^4 + c_6 u^6 + \cdots$$

is the Weierstrass function [1].

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Recall that

$$
\begin{aligned}
c_2 &= \tfrac{1}{20}\, g_2 \\
c_4 &= \tfrac{1}{28}\, g_3 \\
c_6 &= \tfrac{1}{1200}\, g_2\, g_2 \\
c_8 &= \tfrac{3}{6160}\, g_2\, g_3 \\
c_{10} &= \tfrac{1}{10192}\, g_2\, g_2\, g_2 + \tfrac{1}{156000}\, g_3\, g_3 \\
c_{12} &= \tfrac{1}{184800}\, g_2\, g_2\, g_3 \\
&\dots
\end{aligned}
$$

In the paper we consider the analogous uniformization for the hyperelliptic curve.

1. The function $\wp_{2n+1}$

The hyperelliptic curve

$$y^2 = (2n)!^2 x^{2n+1} - g_2 x^{2n-1} - g_3 x^{2n-2} - \cdots - g_{2n+1}$$

has the uniformization

$$
\begin{aligned}
x &= \wp_{2n+1}(u), \\
y &= \wp_{2n+1}^{(2n-1)}(u),
\end{aligned}
$$

where

$$\wp_{2n+1}(u) = u^{-2} + c_2 u^2 + c_4 u^4 + c_6 u^6 + \cdots .$$

The curve

$$y^2 = 576 x^5 - g_2 x^3 - g_3 x^2 - g_4 x - g_5$$

has the uniformization

$$
\begin{aligned}
x &= \wp_5(u), \\
y &= \wp_5^{(3)}(u),
\end{aligned}
$$

where

$$
\begin{aligned}
c_2 &= \tfrac{1}{2880}\, g_2 \\
c_4 &= \tfrac{1}{4032}\, g_3 \\
c_6 &= \tfrac{1}{24883200}\, g_2\, g_2 + \tfrac{1}{8640}\, g_4 \\
c_8 &= \tfrac{1}{42577920}\, g_2\, g_3 + \tfrac{1}{19008}\, g_5 \\
c_{10} &= \tfrac{1}{465813504000}\, g_2\, g_2\, g_2 + \tfrac{1}{161740800}\, g_2\, g_4 + \tfrac{1}{211341312}\, g_3\, g_3 \\
c_{12} &= \tfrac{1}{551809843200}\, g_2\, g_2\, g_3 - \tfrac{1}{1259089920}\, g_2\, g_5 + \tfrac{19}{4006195200}\, g_3\, g_4 \\
&\dots
\end{aligned}
$$

The curve

$$y^2 = 518400 x^7 - g_2 x^5 - g_3 x^4 - g_4 x^3 - g_5 x^2 - g_6 x - g_7$$

has the uniformization

$$\begin{aligned} x &= \wp_7(u), \\ y &= \wp_7^{(5)}(u), \end{aligned}$$

where

$$\begin{aligned}
c_2 &= \tfrac{1}{3628800}\, g_2 \\
c_4 &= \tfrac{1}{3628800}\, g_3 \\
c_6 &= \tfrac{1}{8465264640000}\, g_2\, g_2 + \tfrac{1}{4665600}\, g_4 \\
c_8 &= \tfrac{1}{16094453760000}\, g_2\, g_3 + \tfrac{1}{13305600}\, g_5 \\
c_{10} &= \tfrac{31}{559081292326502400000 0}\, g_2\, g_2\, g_2 + \tfrac{1}{77033908224000}\, g_2\, g_4 + \tfrac{1}{171186462720000}\, g_3\, g_3 + \tfrac{1}{47174400}\, g_6 \\
c_{12} &= \tfrac{1}{261636001950597120000}\, g_2\, g_2\, g_3 + \tfrac{19}{5607767531520000}\, g_2\, g_5 + \tfrac{13}{4588173434880000}\, g_3\, g_4 + \tfrac{1}{140486400}\, g_7 \\
&\ldots
\end{aligned}$$

2. Examples

In the case

$$y^2 = 576x^5 - 1$$

the function $\wp_5$ has the form

$$\wp_5(u) = u^{-2} + \tfrac{1}{19008}u^8 + \tfrac{31}{24869763072}u^{18} + \tfrac{527291}{23300588459533271 04}u^{28} + \cdots .$$

In the case

$$y^2 = 576x^5 - x$$

the function $\wp_5$ has the form

$$\wp_5(u) = u^{-2} + \tfrac{1}{8640}\, u^6 + \tfrac{1}{465315840}\, u^{14} + \tfrac{79}{292102018560000}\, u^{22} + \cdots .$$

Литература

[1] A.Hurwitz, R.Courant. Theory of functions. "Nauka", Moscow, 1968.