Distributions in One Variable

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Contents

Introduction

The little book is a self-contained introduction to the theory of distributions. All sections are suitable for students who have had one or two years of calculus. The manual is based on the book [1]. Necessary facts about measurable and absolutely continuous functions may be taken from the book [2]. In three last sections we need standard facts about linear differential equations with constant coefficients.

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1. Test functions

Let φ be a real function. The set

$$\operatorname{supp}\varphi = \mathrm{Cl}\{x \in \mathbb{R} :\ \varphi(x) \neq 0\}$$

is called the support. Here Cl is the closure in $\mathbb{R}$. The function φ is called finite if the suppurt $\operatorname{supp}\varphi$ is compact. The space of all finite $\varphi \in C^\infty(\mathbb{R})$ is denoted by $C_0^\infty(\mathbb{R})$. Analogously the space of all finite $\varphi \in C(\mathbb{R})$ is denoted by $C_0(\mathbb{R})$. Functions $\varphi \in C_0^\infty(\mathbb{R})$ are called test functions.

Let $\Omega = (a, b)$ be a open interval. Usually we assume that $-\infty < a < b < \infty$. Consider the space

$$C_0^\infty(\Omega) = \{\varphi \in C_0^\infty(\mathbb{R}) :\ \operatorname{supp}\varphi \subset \Omega\}.$$

We say that $\varphi_1, \varphi_2, \ldots \in C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$ if there exists $[\alpha, \beta] \subset \Omega$ such that $\operatorname{supp}\varphi_k \subset [\alpha, \beta]$ for all $k = 1, 2, \ldots$ and

$$\lim_{k\to\infty} \|\varphi^{(i)} - \varphi_k^{(i)}\|_{C[\alpha,\beta]} = 0$$

for all $i = 0, 1, \ldots$.

Let us construct a nonzero test function. Put

$$\lambda = \int_{-1}^{1} \exp\left(\frac{1}{x^2 - 1}\right) dx$$

and define

$$h(x) = \begin{cases} \dfrac{1}{\lambda} \exp\left(\dfrac{1}{x^2 - 1}\right) & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$$

By direct calculation, we see that $h \in C^\infty(\mathbb{R})$ and $\operatorname{supp} h = [-1, 1]$. Hence $h \in C_0^\infty(\mathbb{R})$. Obviously

$$\int_{-1}^{1} h(x)\, dx = 1.$$

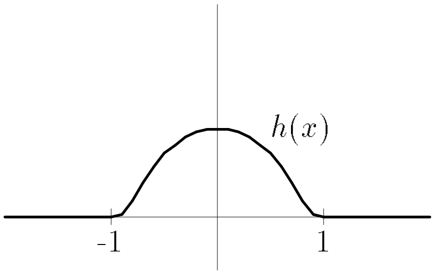
The test function h is called the hat.

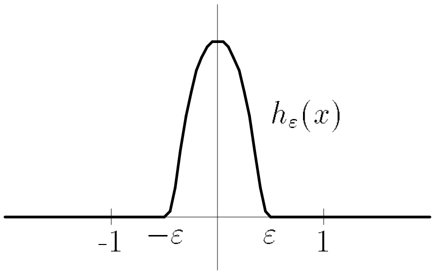
Let $\varepsilon > 0$. The test function

$$h_\varepsilon(x) = \frac{1}{\varepsilon} h\left(\frac{x}{\varepsilon}\right)$$

is called the ε-hat. We see that $\operatorname{supp} h_\varepsilon = [-\varepsilon, \varepsilon]$ and

$$\int_{-\varepsilon}^{\varepsilon} h_\varepsilon(x)\, dx = 1.$$





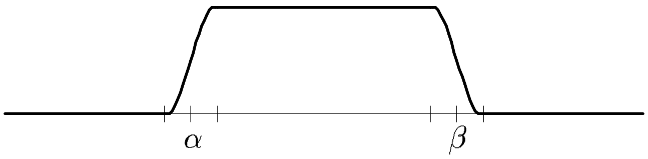
Let $\alpha < \beta$. Then

$$H_\varepsilon^{\alpha,\beta}(x) = \int_\alpha^\beta h_\varepsilon(x-y)\,dy$$

is a test function. Obviously,

$$H_\varepsilon^{\alpha,\beta}(x) = \begin{cases} 1 & \text{for} \quad x \in [\alpha+\varepsilon, \beta-\varepsilon] \\ 0 & \text{for} \quad x \notin [\alpha-\varepsilon, \beta+\varepsilon] \end{cases}$$

This means that the test function $H_\varepsilon^{\alpha,\beta}$ has the form



If $[\alpha,\beta] \subset \Omega$, then $H_\varepsilon^{\alpha,\beta} \in C_0^\infty(\Omega)$. In particular, the space $C_0^\infty(\Omega)$ has nonzero elements.

Consider the space

$$C_0(\Omega) = \{\varphi \in C_0(\mathbb{R}) : \ \mathrm{supp}\varphi \subset \Omega\}.$$

We say that $\varphi_1, \varphi_2, \ldots \in C_0(\Omega)$ converges to $\varphi \in C_0(\Omega)$ if there exists $[\alpha,\beta] \subset \Omega$ such that $\mathrm{supp}\varphi_k \subset [\alpha,\beta]$ for all $k = 1, 2, \ldots$ and

$$\lim_{k\to\infty} \|\varphi - \varphi_k\|_{C[\alpha,\beta]} = 0.$$

Let $*$ be the convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

If $\varphi \in C_0(\Omega)$, then $h_\varepsilon * \varphi \in C_0^\infty(\Omega)$ for all sufficiently small $\varepsilon > 0$.

Theorem 1 *If $\varphi \in C_0(\Omega)$, then*

$$\varphi = \lim_{\varepsilon\to 0} \varphi * h_\varepsilon$$

in the space $C_0(\Omega)$.

Proof. There exists $[\alpha,\beta] \subset \Omega$ such that $\operatorname{supp}\varphi \subset [\alpha,\beta]$ and $\operatorname{supp}(\varphi * h_\varepsilon) \subset [\alpha,\beta]$ for all sufficiently small ε. We see

$$\mid \varphi(x) - (\varphi * h_\varepsilon)(x) \mid = \left| \int_{-\infty}^{\infty} [\varphi(x) - \varphi(x-y)] h_\varepsilon(y)\, dy \right| \leq \sup_{x\in \mathrm{R}} \sup_{|y|\leq\varepsilon} \mid \varphi(x) - \varphi(x-y) \mid$$

Since φ is uniformly continuous, we have

$$\lim_{\varepsilon\to 0} \|\varphi - \varphi * h_\varepsilon\|_{C[\alpha,\beta]} = 0.$$

As was to be proved. •

Corollary. The space $C_0^\infty(\Omega)$ is dence in the space $C_0(\Omega)$.

2. Distributions

A linear continuous functional on the space $C_0^\infty(\Omega)$ is called a distribution. The space of all distributions is denoted by $\mathcal{D}'(\Omega)$. The value of the distribution $\Lambda \in \mathcal{D}'(\Omega)$ at the test function $\varphi \in C_0^\infty(\Omega)$ is denoted by (Λ, φ). The continuity of $\Lambda \in \mathcal{D}'(\Omega)$ means that

$$(\Lambda, \varphi) = \lim_{k\to\infty} (\Lambda, \varphi_k)$$

whenever

$$\varphi = \lim_{k\to\infty} \varphi_k.$$

In other words Λ is sequentially continuous.

We say that $\Lambda_1, \Lambda_2, \ldots \in \mathcal{D}'(\Omega)$ converges to $\Lambda \in \mathcal{D}'(\Omega)$ if

$$(\Lambda, \varphi) = \lim_{k\to\infty} (\Lambda_k, \varphi)$$

for any $\varphi \in C_0^\infty(\Omega)$.

Let f be a measurable function on the interval Ω. The function f is called locally integrable if

$$\int_\alpha^\beta |f(x)|\, dx < \infty$$

for any $[\alpha,\beta] \subset \Omega$. Let $L_1(\Omega)^{\mathrm{loc}}$ be the space of all locally integrable functions. We say that $f_1, f_2, \ldots \in L_1(\Omega)^{\mathrm{loc}}$ converges to $f \in L_1(\Omega)^{\mathrm{loc}}$ if

$$\lim_{k\to\infty} \|f - f_k\|_{L_1[\alpha,\beta]} = 0$$

for any $[\alpha,\beta] \subset \Omega$.

Let $f \in L_1(\Omega)^{\mathrm{loc}}$. Define $\Lambda_f \in \mathcal{D}'(\Omega)$ by the rule

$$(\Lambda_f, \varphi) = \int_\Omega f(x)\varphi(x)\, dx,$$

where $\varphi \in C_0^\infty(\Omega)$. A distribution $\Lambda \in \mathcal{D}'(\Omega)$ is called regular if $\Lambda = \Lambda_f$ for some $f \in L_1(\Omega)^{\mathrm{loc}}$.

Theorem 2 *Let $f \in L_1(\Omega)^{\mathrm{loc}}$. If $f \neq 0$, then $\Lambda_f \neq 0$.*

Proof. Assume that $f \neq 0$. Put

$$\begin{aligned} S_\delta^+ &= \{x \in \Omega : \ f(x) \geq +\delta\}, \\ S_\delta^- &= \{x \in \Omega : \ f(x) \leq -\delta\}. \end{aligned}$$

We see that $\mathrm{mes}\{S_\delta^+\} > 0$ or $\mathrm{mes}\{S_\delta^-\} > 0$ for some $\delta > 0$. Suppose that

$$\mathrm{mes}\{S_\delta^+\} = 2\mu > 0.$$

Then there exists a compact set $A \subset S_\delta^+$ such that $\mathrm{mes}\{A\} \geq \mu$. The compact set A is representable in the form

$$A = \bigcap_{n=1}^{\infty} U^n,$$

where open sets $\Omega \supset U^1 \supset U^2 \supset \ldots$ are such that

$$\mathrm{mes}\{U^n \setminus A\} \leq \frac{1}{n}.$$

We may assume that U^n consists of a finite number of open intervals

$$U^n = \bigcup_{k=1}^{m_n} (\alpha_k^n, \beta_k^n)$$

where $\beta_k^n < \alpha_{k+1}^n$ for all $k = 1, \ldots, m_n - 1$. Let

$$\varphi_\varepsilon^n = \sum_{k=1}^{m_n} H_\varepsilon^{\alpha_k^n - \varepsilon, \beta_k^n + \varepsilon}.$$

We see that

$$\int\limits_\Omega f(x)\varphi_\varepsilon^n(x)\, dx \geq \int\limits_A f(x)\, dx - \int\limits_{U^n \setminus A} \mid f(x) \mid \, dx - \left(\sum_{k=1}^{m_n} \int\limits_{\alpha_k^n - 2\varepsilon}^{\alpha_k^n} \mid f(x) \mid \, dx + \sum_{k=1}^{m_n} \int\limits_{\beta_k^n}^{\beta_k^n + 2\varepsilon} \mid f(x) \mid \, dx \right).$$

We note that

$$\int\limits_A f(x)\, dx \geq \delta\mu > 0.$$

Obviously,

$$\lim_{n \to \infty} \int\limits_{U^n \setminus A} \mid f(x) \mid \, dx = 0$$

and

$$\lim_{\varepsilon \to 0} \left(\sum_{k=1}^{m_n} \int\limits_{\alpha_k^n - 2\varepsilon}^{\alpha_k^n} \mid f(x) \mid \, dx + \sum_{k=1}^{m_n} \int\limits_{\beta_k^n}^{\beta_k^n + 2\varepsilon} \mid f(x) \mid \, dx \right) = 0.$$

Hence

$$\int_{\Omega} f(x)\varphi_{\varepsilon}^{n}(x)\,dx > 0$$

for some n and ε. This means that $\Lambda_f \neq 0$. •

By Theorem 2, the mapping

$$L_1(\Omega)^{\mathrm{loc}} \longrightarrow \mathcal{D}'(\Omega)$$

such that

$$f \longmapsto \Lambda_f$$

is a sequentially continuous embedding.

Let $x_0 \in \Omega$. The distribution $\delta_{x_0} \in \mathcal{D}'(\Omega)$ such that

$$(\delta_{x_0}, \varphi) = \varphi(x_0)$$

for all $\varphi \in C_0^{\infty}(\Omega)$ is called the delta-function.

Theorem 3 *The delta-function $\delta_0 \in \mathcal{D}'(\mathbb{R})$ is not regular.*

Proof. Let $f \in L_1(\mathbb{R})^{\mathrm{loc}}$ be such that

$$(\delta_0, \varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

for all $\varphi \in C_0^{\infty}(\mathbb{R})$. Put

$$\varphi_{\varepsilon}(x) = h(x/\varepsilon).$$

On the one hand, we have

$$(\delta_0, \varphi_{\varepsilon}) = \varphi_{\varepsilon}(0) = h(0) > 0.$$

On the other hand, we have

$$\left|\int_{-\infty}^{\infty} f(x)\varphi_{\varepsilon}(x)\,dx\right| = \left|\int_{-\varepsilon}^{\varepsilon} f(x)h(x/\varepsilon)\,dx\right| \leq h(0)\int_{-\varepsilon}^{\varepsilon} \mid f(x) \mid\, dx.$$

Since

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \mid f(x) \mid\, dx = 0,$$

we have the contradiction. •

Theorem 4 *We have*

$$\delta_0 = \lim_{\varepsilon\to 0} h_{\varepsilon}$$

in the space $\mathcal{D}'(\mathbb{R})$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}
|\,(\delta_0, \varphi) - (h_\varepsilon, \varphi)\,| &= \left| \varphi(0) - \int_{\infty}^{\infty} h_\varepsilon(x)\varphi(x)\,dx \right| \\
&= \left| \int_{\infty}^{\infty} h_\varepsilon(x)(\varphi(0) - \varphi(x))\,dx \right| \\
&\leq \int_{\infty}^{\infty} h_\varepsilon(x)\,|\,\varphi(0) - \varphi(x)\,|\ dx \\
&= \int_{\infty}^{\infty} h_\varepsilon(x) \left| \int_0^x \varphi'(t)\,dt \right| dx \\
&\leq \varepsilon \|\varphi'\|_{C(\mathrm{R})}.
\end{aligned}$$

As was to be proved. •

3. Distributions with compact supports

We say that $\Lambda \in \mathcal{D}'(\Omega)$ is equal to zero at $x \in \Omega$ if there exists $\varepsilon > 0$ such that $(\Lambda, \varphi) = 0$ for all $\varphi \in C_0^\infty(\Omega)$ such that $\mathrm{supp}\varphi \subset [x - \varepsilon, x + \varepsilon]$. For example, the delta-function δ_{x_0} is equal to zero outside of x_0.

Let $\mathcal{O}(\Lambda)$ be the set of all points $x \in \Omega$ at which Λ is equal to zero. Then $\mathcal{O}(\Lambda)$ is open in Ω. Hence $\Omega \setminus \mathcal{O}(\Lambda)$ is closed in Ω. The set $\mathrm{supp}\,\Lambda = \Omega \setminus \mathcal{O}(\Lambda)$ is called the support of Λ. The space of distributions $\Lambda \in \mathcal{D}'(\Omega)$ with compact supports is denoted by $\mathcal{E}'(\Omega)$. We see that $\mathrm{supp}\delta_{x_0} = \{x_0\}$. Hence $\delta_{x_0} \in \mathcal{E}'(\Omega)$ for all $x_0 \in \Omega$.

Theorem 5 *Let $\Lambda \in \mathcal{D}'(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. If $\mathrm{supp}\Lambda \cap \mathrm{supp}\varphi = \emptyset$, then $(\Lambda, \varphi) = 0$.*

Proof. Consider the test function

$$\psi_\varepsilon^k(x) = \left(\sum_{i=-\infty}^{\infty} h_\varepsilon(x - \varepsilon i) \right)^{-1} h_\varepsilon(x - \varepsilon k).$$

Note that $\mathrm{supp}\,\psi_\varepsilon^k = [\varepsilon(k-1), \varepsilon(k+1)]$. Since

$$\sum_{k=-\infty}^{\infty} \psi_\varepsilon^k(x) = 1,$$

we have

$$\varphi = \sum_{k=-\infty}^{\infty} \psi_\varepsilon^k \varphi.$$

Note that the right-hand sum consists of a finite number of nonzero terms. Since suppφ is compact, since suppΛ is closed in Ω, and since supp$\Lambda \cap$ supp$\varphi = \emptyset$, we have

$$(\Lambda, \varphi) = \sum_{k=-\infty}^{\infty} (\Lambda, \psi_\varepsilon^k \varphi) = 0,$$

where $\varepsilon > 0$ is sufficiently small. •

Consider the space $C^\infty(\Omega)$. We say that $\varphi_1, \varphi_2, \ldots \in C^\infty(\Omega)$ converges to $\varphi \in C^\infty(\Omega)$ if

$$\lim_{k\to\infty} \|\varphi^{(i)} - \varphi_k^{(i)}\|_{C[\alpha,\beta]} = 0$$

for any $[\alpha, \beta] \subset \Omega$ and for any $i = 0, 1, \ldots$. We see that $C_0^\infty(\Omega)$ is dense in $C^\infty(\Omega)$.

Theorem 6 *A distribution $\Lambda \in \mathcal{D}'(\Omega)$ has a compact support if and only if Λ may be continuously extended from $C_0^\infty(\Omega)$ onto $C^\infty(\Omega)$.*

Proof. Necessity. Let $\Lambda \in \mathcal{D}'(\Omega)$ has a compact support. Then there exist an open set U such that supp$\Lambda \subset U \subset \Omega$ and a test function $\psi \in C_0^\infty(\Omega)$ such that $\psi(x) = 1$ for all $x \in U$. Define a functional $\tilde{\Lambda}$ on the space $C^\infty(\Omega)$ by the rule

$$(\tilde{\Lambda}, \varphi) = (\Lambda, \psi\varphi),$$

where $\varphi \in C^\infty(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$. Since supp$\Lambda \cap$ supp$[(1-\psi)\varphi] = \emptyset$, we have

$$(\tilde{\Lambda}, \varphi) = (\Lambda, \psi\varphi) = (\Lambda, \psi\varphi) + (\Lambda, (1-\psi)\varphi) = (\Lambda, \varphi).$$

Sufficiency. Let Λ be continuously extended from $C_0^\infty(\Omega)$ onto $C^\infty(\Omega)$, and let $\tilde{\Lambda}$ be the extension. Assume that suppΛ is not compact. Then there exists a sequence $x_1, x_2, \ldots \in$ suppΛ such that

$$\lim_{k\to\infty} x_k = a$$

or

$$\lim_{k\to\infty} x_k = b.$$

Let $0 < \varepsilon_k < 1$ be such that $[x_k - \varepsilon_k, x_k + \varepsilon_k] \subset \Omega$. Then let $\psi_k \in C_0^\infty(\Omega)$ be such that supp$\psi_k \subset [x_k - \varepsilon_k, x_k + \varepsilon_k]$ and $(\Lambda, \psi_k) = 1$. We see that

$$\lim_{k\to\infty} \psi_k = 0$$

in in the space $C^\infty(\Omega)$. Hence,

$$\lim_{k\to\infty} (\tilde{\Lambda}, \psi_k) = 0.$$

Since

$$(\tilde{\Lambda}, \psi_k) = (\Lambda, \psi_k) = 1$$

for all $k = 1, 2, \ldots$, we have the contradiction. Thus suppΛ is compact. •

4. Positive distributions

Let $f \in L_1(\Omega)^{\mathrm{loc}}$. By Theorem 2, we see that $f(x) \geq 0$ for almost all $x \in \Omega$ if and only if

$$\int_\Omega f(x)\varphi(x)\,dx \geq 0$$

for all positive $\varphi \in C_0^\infty(\Omega)$. This leads to the following definition.

A distribution $\Lambda \in \mathcal{D}'(\Omega)$ is called positive if

$$(\Lambda, \varphi) \geq 0$$

for all positive $\varphi \in C_0^\infty(\Omega)$. To denote the fact that $\Lambda \in \mathcal{D}'(\Omega)$ is positive, we write $\Lambda \geq 0$.

The delta-function is positive. Indeed, if $x_0 \in \Omega$, then

$$(\delta_{x_0}, \varphi) = \varphi(x_0) \geq 0$$

for all positive $\varphi \in C_0^\infty(\Omega)$. Hence $\delta_{x_0} \geq 0$.

Let $x_1, \dots, x_k \in \Omega$ and $\lambda_1 \geq 0, \dots, \lambda_k \geq 0$. Then

$$\Lambda = \lambda_1 \delta_{x_1} + \dots + \lambda_k \delta_{x_k}$$

is positive.

Theorem 7 *Let $\Lambda \in \mathcal{D}'(\Omega)$ be positive. Given $[\alpha, \beta] \subset \Omega$, there exists $C > 0$ such that*

$$|\,(\Lambda, \varphi)\,| \leq C \|\varphi\|_{C[\alpha,\beta]}$$

for all $\varphi \in C_0^\infty(\alpha, \beta)$.

Proof. Let $\psi \in C_0^\infty(\Omega)$ be positive, and let $\psi(x) = 1$ for all $x \in [\alpha, \beta]$. If $\varphi \in C_0^\infty(\alpha, \beta)$, then

$$-\|\varphi\|_{C[\alpha,\beta]}\psi \leq \varphi \leq \|\varphi\|_{C[\alpha,\beta]}\psi.$$

Hence,

$$-\|\varphi\|_{C[\alpha,\beta]}(\Lambda, \psi) \leq (\Lambda, \varphi) \leq \|\varphi\|_{C[\alpha,\beta]}(\Lambda, \psi).$$

This means that

$$|\,(\Lambda, \varphi)\,| \leq C \|\varphi\|_{C[\alpha,\beta]}$$

for all $\varphi \in C_0^\infty(\alpha, \beta)$ with $C = (\Lambda, \psi)$. •

We say that $\Lambda \in \mathcal{D}'(\Omega)$ is a zero order distribution if given $[\alpha, \beta] \subset \Omega$, there exists $C \geq 0$ such that

$$|\,(\Lambda, \varphi)\,| \leq C \|\varphi\|_{C[\alpha,\beta]}$$

for all $\varphi \in C_0^\infty(\alpha, \beta)$. By Theorem 7, if $\Lambda \geq 0$, then Λ is a zero order distribution.

Theorem 8 *Let Λ be a linear continuous functional on $C_0(\Omega)$, then Λ is a zero order distribution.*

Proof. Since $C_0^\infty(\Omega) \subset C_0(\Omega)$, we have $\Lambda \in \mathcal{D}'(\Omega)$. Assume that Λ is not a zero order distribution. Hence there exists $[\alpha, \beta] \subset \Omega$ such that for each $k = 1, 2, \dots$ we can find $\varphi_k \in C_0^\infty(\alpha, \beta)$ satisfying

$$| (\Lambda, \varphi_k) | > k \|\varphi_k\|_{C[\alpha,\beta]}.$$

Let

$$\psi_k = \frac{1}{k} \|\varphi_k\|^{-1}_{C[\alpha,\beta]} \varphi_k.$$

On the one hand, we see that

$$\lim_{k\to\infty} \psi_k = 0$$

in the space $C_0(\Omega)$. Hence

$$\lim_{k\to\infty} (\Lambda, \psi_k) = 0.$$

On the other hand, we see that

$$| (\Lambda, \psi_k) | = \frac{1}{k} \|\varphi_k\|^{-1}_{C[\alpha,\beta]} \, | (\Lambda, \varphi_k) | > 1.$$

The contradiction shows that Λ is a zero order distribution. •

Theorem 9 *Let $\Lambda \in \mathcal{D}'(\Omega)$ be a zero order distribution. Then Λ can be continuously extended from $C_0^\infty(\Omega)$ onto $C_0(\Omega)$.*

Proof. The unique continuous extension $\tilde{\Lambda}$ is defined by the rule

$$(\tilde{\Lambda}, \varphi) = \lim_{\varepsilon\to 0} (\Lambda, \varphi * h_\varepsilon),$$

where $\varphi \in C_0(\Omega)$. •

In the following statement we need the Stieltjes integral and the Riesz theorem about linear continuous functionals on the space of continuous functions. Assume that any increasing function $f(x)$ is continuous from the right. This means, that

$$f(x_0) = f(x_0 + 0) = \lim_{\varepsilon \searrow 0} f(x_0 + \varepsilon)$$

for all x_0.

Theorem 10 *Let $\Lambda \in \mathcal{D}'(\Omega)$ be positive. Given $x_0 \in \Omega$, there exists a unique increasing function f_Λ such that $f_\Lambda(x_0) = 0$ and*

$$(\Lambda, \varphi) = \int_\Omega \varphi(x) \, df_\Lambda(x)$$

for all $\varphi \in C_0^\infty(\Omega)$.

Proof. We may assume that Λ is extended from $C_0^\infty(\Omega)$ onto $C_0(\Omega)$. Let $x_0 \in [\alpha, \beta] \subset \Omega$. Consider the space

$$S[\alpha,\beta] = \{\varphi \in C[\alpha,\beta]: \ \varphi(\alpha) = \varphi(\beta) = 0\}.$$

We may assume that $\varphi \in S[\alpha,\beta]$ is equal to zero outside of $[\alpha,\beta]$. Since Λ is positive on $S[\alpha,\beta]$, there exists a unique increasing function $f_\Lambda^{\alpha,\beta}$ such that $f_\Lambda^{\alpha,\beta}(x_0) = 0$ and

$$(\Lambda, \varphi) = \int_\alpha^\beta \varphi(x)\, df_\Lambda^{\alpha,\beta}(x)$$

for all $\varphi \in S[\alpha,\beta]$. If $[\alpha,\beta] \subset [\gamma,\tau] \subset \Omega$, then $f_\Lambda^{\gamma,\tau}|_{[\alpha,\beta]} = f_\Lambda^{\alpha,\beta}$. Therefore there exists a unique increasing function f_Λ such that

$$f_\Lambda(x_0) = 0$$

and

$$(\Lambda, \varphi) = \int_\Omega \varphi(x)\, df_\Lambda(x)$$

for all $\varphi \in C_0(\Omega)$. $\bullet$

Corollary. Let $\Lambda \in \mathcal{D}'(\Omega)$ be positive. There exists a unique Borel measure μ_Λ such that

$$(\Lambda, \varphi) = \int_\Omega \varphi(x)\, d\mu_\Lambda(x)$$

for all $\varphi \in C_0^\infty(\Omega)$. Indeed, we put

$$\mu_\Lambda\{(\alpha,\beta]\} = f_\Lambda(\beta) - f_\Lambda(\alpha),$$

where $[\alpha,\beta] \subset \Omega$. Then we define the measure μ_Λ on the σ-algebra of Borel sets by the standard extension. In particular, we have

$$\mu_\Lambda\{[\alpha,\beta]\} = f_\Lambda(\beta) - f_\Lambda(\alpha - 0) \quad \text{and} \quad \mu_\Lambda\{(\alpha,\beta)\} = f_\Lambda(\beta - 0) - f_\Lambda(\alpha)$$

5. The derivative

Consider $f \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Integrating by parts, we have

$$\int_\Omega f'(x)\varphi(x)\, dx = -\int_\Omega f(x)\varphi'(x)\, dx.$$

The right-hand side integral has sense for any $f \in L_1(\Omega)^{\mathrm{loc}}$. This leads to the following definition.

Let $\Lambda \in \mathcal{D}'(\Omega)$. Define the distribution $\Lambda' \in \mathcal{D}'(\Omega)$ by the rule

$$(\Lambda', \varphi) = -(\Lambda, \varphi'),$$

where $\varphi \in C_0^\infty(\Omega)$. The distribution Λ' is called the derivative of the distribution Λ.

Theorem 11 *If $f \in C^1(\Omega)$, then $(\Lambda_f)' = \Lambda_{f'}$.*

Proof. Indeed, let $\varphi \in C_0^\infty(\Omega)$. Then

$$((\Lambda_f)', \varphi) = -(\Lambda_f, \varphi') = -\int_\Omega f(x)\varphi'(x)\,dx = \int_\Omega f'(x)\varphi(x)\,dx = (\Lambda_{f'}, \varphi).$$

As was to be proved. •

A function f is called absolutly continuous on $[\alpha, \beta]$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^n \mid f(\beta_k) - f(\alpha_k) \mid < \varepsilon$$

whenever $\alpha \leq \alpha_1 < \beta_1 < \ldots < \alpha_n < \beta_n \leq \beta$ satisfy

$$\sum_{k=1}^n (\beta_k - \alpha_k) < \delta.$$

A function f is called locally absolutly continuous on Ω if f is absolutly continuous on any $[\alpha, \beta] \subset \Omega$. If f is locally absolutly continuous on Ω, then the classic derivative $f'(x)$ exists for almost all $x \in \Omega$ and $f' \in L_1(\Omega)^{\text{loc}}$.

Theorem 12 *If f is locally absolutly continuous on Ω, then $(\Lambda_f)' = \Lambda_{f'}$.*

Proof. Indeed, let $\varphi \in C_0^\infty(\Omega)$ and $\text{supp}\varphi \subset [\alpha, \beta] \subset \Omega$. Then

$$-\int_\alpha^\beta f(x)\varphi'(x)\,dx = -\int_\alpha^\beta f(x)\,d\varphi(x) = \int_\alpha^\beta \varphi(x)\,df(x) = \int_\alpha^\beta f'(x)\varphi(x)\,dx.$$

Hence,

$$((\Lambda_f)', \varphi) = -\int_\alpha^\beta f(x)\varphi'(x)\,dx = \int_\alpha^\beta f'(x)\varphi(x)\,dx = (\Lambda_{f'}, \varphi).$$

As was to be proved. •

Consider the Heaviside function

$$\Theta(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}.$$

Theorem 13 *We have $\Theta' = \delta_0$.*

Proof. Indeed, let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$(\Theta', \varphi) = -(\Theta, \varphi') = -\int_{-\infty}^{\infty} \Theta(x)\varphi'(x)\,dx = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) = (\delta_0, \varphi).$$

As was to be proved. •

Theorem 14 *If $f \in C^1(\mathbb{R})$, then $(\Theta f)' = f(0)\delta_0 + \Theta f'$.*

Proof. Indeed, let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned} ((\Theta f)', \varphi) &= -(\Theta f, \varphi') \\ &= -\int_{-\infty}^{\infty} \Theta(x) f(x) \varphi'(x)\, dx = -\int_0^{\infty} f(x)\varphi'(x)\, dx = -\int_0^{\infty} f(x)\, d\varphi(x) \\ &= f(0)\varphi(0) + \int_0^{\infty} f'(x)\varphi(x)\, dx = f(0)\varphi(0) + \int_{-\infty}^{\infty} \Theta(x) f'(x)\varphi(x)\, dx \\ &= (f(0)\delta_0 + \Theta f', \varphi). \end{aligned}$$

As was to be proved. •

Let

$$x_+^k = \begin{cases} x^k & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}.$$

We see that $x_+^k = \Theta x^k$. Hence

$$\frac{d^l}{dx^l} \frac{x_+^{l-1}}{(l-1)!} = \delta_0.$$

Let $f \in C^\infty(\Omega)$ and $\Lambda \in \mathcal{D}'(\Omega)$. The product $f\Lambda \in \mathcal{D}'(\Omega)$ is defined by the rule

$$(f\Lambda, \varphi) = (\Lambda, f\varphi),$$

where $\varphi \in C_0^\infty(\Omega)$.

Theorem 15 *Let $f \in C^\infty(\Omega)$ and $\Lambda \in \mathcal{D}'(\Omega)$. Then $(f\Lambda)' = f\Lambda' + f'\Lambda$.*

Proof. Indeed, let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned} ((f\Lambda)', \varphi) &= -(f\Lambda, \varphi') = -(\Lambda, f\varphi') \\ &= -(\Lambda, (f\varphi)' - f'\varphi) = -(\Lambda, (f\varphi)') + (\Lambda, f'\varphi) \\ &= (\Lambda', f\varphi) + (\Lambda, f'\varphi) = (f\Lambda', \varphi) + (f'\Lambda, \varphi) \\ &= (f\Lambda' + f'\Lambda, \varphi) \end{aligned}$$

As was to be proved. •

6. The Integral

Theorem 16 *Let $u \in \mathcal{D}'(\Omega)$ be such that $u' = 0$. Then u is a constant function.*

Proof. Let $\psi \in C_0^\infty(\Omega)$ be such that

$$\int_a^b \psi(x)\,dx = 1.$$

Define the mapping

$$\Phi_\psi :\ C_0^\infty(\Omega) \longrightarrow C_0^\infty(\Omega)$$

by the rule

$$\Phi_\psi(\varphi)(x) = \int_a^x \varphi(y)\,dy - \int_a^x \psi(y)\,dy \int_a^b \varphi(y)\,dy.$$

Put $C = (u, \psi)$. Let $\varphi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned}
(u, \varphi) &= \left(u, \Phi_\psi(\varphi)' + \psi \int_a^b \varphi(y)\,dy\right) \\
&= (u, \Phi_\psi(\varphi)') + (u, \psi) \int_a^b \varphi(y)\,dy \\
&= -(u', \Phi_\psi(\varphi)) + C \int_a^b \varphi(y)\,dy \\
&= \int_a^b C\varphi(y)\,dy \\
&= (C, \varphi).
\end{aligned}$$

Therefore $u = C$. •

Theorem 17 *Let $\psi \in C_0^\infty(\Omega)$ be such that*

$$\int_a^b \psi(x)\,dx = 1.$$

Given $\Lambda \in \mathcal{D}'(\Omega)$, there exists a unique $u \in \mathcal{D}'(\Omega)$ such that $u' = \Lambda$ and $(u, \psi) = 0$.

Proof. Define the distribution $u \in \mathcal{D}'(\Omega)$ by the rule

$$(u, \varphi) = -(\Lambda, \Phi_\psi(\varphi)),$$

where again

$$\Phi_\psi(\varphi)(x) = \int\limits_a^x \varphi(y)\,dy - \int\limits_a^x \psi(y)\,dy \int\limits_a^b \varphi(y)\,dy.$$

Since

$$\int\limits_a^b \varphi'(y)\,dy = 0,$$

we have $\Phi_\psi(\varphi') = \varphi$. Therefore

$$(u', \varphi) = -(u, \varphi') = (\Lambda, \Phi_\psi(\varphi')) = (\Lambda, \varphi).$$

This means that $u' = \Lambda$. Since $\Phi_\psi(\psi) = 0$, we have $(u, \psi) = 0$. By Theorem 16, we see that u is a unique distribution satisfying the conditions. •

Corollary. There exists a linear continuous operator

$$I:\ \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

such that

$$\frac{d}{dx} I(\Lambda) = \Lambda$$

for all $\Lambda \in \mathcal{D}'(\Omega)$.

7. The convolution of functions

We say that the family of k functions and distributions is admissible if at least $k-1$ of them have compact supports. Usually we have $k = 2$ or $k = 3$.

Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R})$ be admissible. The convolution of φ_1 and φ_2 is define by the rule

$$\varphi_1 * \varphi_2(x) = \int\limits_{-\infty}^{\infty} \varphi_1(x-y)\varphi_2(y)\,dy.$$

By the identity

$$\int\limits_{-\infty}^{\infty} \varphi_1(x-y)\varphi_2(y)\,dy = \int\limits_{-\infty}^{\infty} \varphi_1(y)\varphi_2(x-y)\,dy,$$

we have

$$\varphi_1 * \varphi_2 = \varphi_2 * \varphi_1.$$

This means that the convolution is commutative.

By differentiation under the integral, we have

$$(\varphi_1 * \varphi_2)' = \varphi_1' * \varphi_2 = \varphi_1 * \varphi_2'.$$

In particular, we have

$$\varphi_1 * \varphi_2 \in C^\infty(\mathbb{R}).$$

Let us check that

$$\mathrm{supp}(\varphi_1 * \varphi_2) \subset \mathrm{supp}\varphi_1 + \mathrm{supp}\varphi_2.$$

Indeed, let $z_0 \in \mathbb{R}$ be such that $\varphi_1 * \varphi_2(z_0) \neq 0$. Since

$$\int_{-\infty}^{\infty} \varphi_1(z_0 - y)\varphi_2(y)\, dy \neq 0,$$

there exists $y_0 \in \mathbb{R}$ such that $\varphi_1(z_0 - y_0) \neq 0$ and $\varphi_2(y_0) \neq 0$. Let $x_0 = z_0 - y_0$. We see that $x_0 \in \mathrm{supp}\varphi_1$ and $y_0 \in \mathrm{supp}\varphi_2$. Since $\mathrm{supp}\varphi_1$ or $\mathrm{supp}\varphi_2$ is compact, the set $\mathrm{supp}\varphi_1 + \mathrm{supp}\varphi_2$ is closed. Therefore $\mathrm{supp}(\varphi_1 * \varphi_2) \subset \mathrm{supp}\varphi_1 + \mathrm{supp}\varphi_2$.

In particular, if $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$, then $\varphi_1 * \varphi_2 \in C_0^\infty(\mathbb{R})$. Moreover, by the identity

$$\int_{-\infty}^{\infty} \varphi_1 * \varphi_2(x)\varphi(x)\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_1(x)\varphi_2(y)\varphi(x+y)\, dxdy,$$

where $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} \min\,\mathrm{supp}(\varphi_1 * \varphi_2) &= \min\,\mathrm{supp}\varphi_1 + \min\,\mathrm{supp}\varphi_2, \\ \max\,\mathrm{supp}(\varphi_1 * \varphi_2) &= \max\,\mathrm{supp}\varphi_1 + \max\,\mathrm{supp}\varphi_2. \end{aligned}$$

Let $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(\mathbb{R})$ be admissible, then

$$\varphi_1 * (\varphi_2 * \varphi_3) = (\varphi_1 * \varphi_2) * \varphi_3.$$

This means that the convolution is associative.

Theorem 18 *Let $\varphi \in C^\infty(\mathbb{R})$, then*

$$\varphi = \lim_{\varepsilon \to 0} \varphi * h_\varepsilon$$

in the space $C^\infty(\mathbb{R})$.

Proof. Indeed, let $\alpha < \beta$, and let $x \in [\alpha, \beta]$, then

$$\begin{aligned} \left|\varphi^{(k)}(x) - (\varphi * h_\varepsilon)^{(k)}(x)\right| &= \left|\varphi^{(k)}(x) - \varphi^{(k)} * h_\varepsilon(x)\right| \\ &= \left|\varphi^{(k)}(x) - \int_{-\infty}^{\infty} \varphi^{(k)}(x-y)h_\varepsilon(y)\, dy\right| \\ &= \left|\int_{-\varepsilon}^{\varepsilon} \left(\varphi^{(k)}(x) - \varphi^{(k)}(x-y)\right) h_\varepsilon(y)\, dy\right| \\ &\leq \varepsilon\|\varphi^{(k+1)}\|_{C[\alpha-\varepsilon,\beta+\varepsilon]}\|. \end{aligned}$$

Therefore

$$\|\varphi^{(k)} - (\varphi * h_\varepsilon)^{(k)}\|_{C[\alpha,\beta]} \le \varepsilon \|\varphi^{(k+1)}\|_{C[\alpha-\varepsilon,\beta+\varepsilon]}$$

for all $k = 0, 1, \ldots$. •

Corollary. Let $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\varphi = \lim_{\varepsilon \to 0} \varphi * h_\varepsilon$$

in the space $C_0^\infty(\mathbb{R})$.

8. The convolution of distributions and functions

Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ be admissible. By definition, this means that $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R})$ or $\Lambda \in \mathcal{E}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$. In the second case we assume that Λ is extended from $C_0^\infty(\mathbb{R})$ onto $C^\infty(\mathbb{R})$.

The convolution of Λ and φ is defined by the rule

$$\Lambda * \varphi(x) = (\Lambda, \varphi(x - \cdot\,)),$$

where $x \in \mathbb{R}$. In particular, the convolution $\Lambda * \varphi$ is a real function.

We shall use the notation

$$\overset{\vee}{\varphi}(x) = \varphi(-x).$$

Note that

$$(\Lambda, \varphi) = \left(\Lambda * \overset{\vee}{\varphi}\right)(0).$$

Theorem 19 *Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ be admissible. Then $(\Lambda * \varphi)' = \Lambda * \varphi'$.*

Proof. Let $x_0 \in \mathbb{R}$. If $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\varphi(x_0 - \cdot\,) = \lim_{\Delta x \to 0} \frac{\varphi(x_0 + \Delta x - \cdot\,) - \varphi(x_0 - \cdot\,)}{\Delta x}$$

in the space $C_0^\infty(\mathbb{R})$. If $\varphi \in C^\infty(\mathbb{R})$, then analogously

$$\varphi(x_0 - \cdot\,) = \lim_{\Delta x \to 0} \frac{\varphi(x_0 + \Delta x - \cdot\,) - \varphi(x_0 - \cdot\,)}{\Delta x}$$

in the space $C^\infty(\mathbb{R})$. Since

$$\frac{\Lambda * \varphi(x_0 + \Delta x) - \Lambda * \varphi(x_0)}{\Delta x} = \left(\Lambda, \frac{\varphi(x_0 + \Delta x - \cdot\,) - \varphi(x_0 - \cdot\,)}{\Delta x}\right),$$

we have

$$(\Lambda * \varphi)'(x_0) = \Lambda * \varphi'(x_0).$$

As was to be proved. •

Corollary. We have $\Lambda * \varphi \in C^\infty(\mathbb{R})$.

Theorem 20 *Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ be admissible. Then* $\mathrm{supp}(\Lambda * \varphi) \subset \mathrm{supp}\Lambda + \mathrm{supp}\varphi$.

Proof. Let $\psi(x) = 1$ on the whole $\mathbb{R}$ if $\mathrm{supp}\Lambda$ is unbounded, and let $\psi(x) = 1$ on a bounded neighborhood of $\mathrm{supp}\Lambda$ if $\mathrm{supp}\Lambda$ is bounded. Consider $z_0 \in \mathbb{R}$ such that $\Lambda * \varphi(z_0) \neq 0$. Since

$$\Lambda * \varphi(z_0) = (\Lambda, \varphi(z_0 - \cdot\,)) = (\Lambda, \psi(\cdot)\varphi(z_0 - \cdot\,))$$

and $\psi(\cdot)\varphi(z_0 - \cdot\,) \in C_0^\infty(\mathbb{R})$, we have

$$\mathrm{supp}\Lambda \cap \mathrm{supp}[\psi(\cdot)\varphi(z_0 - \cdot\,)] \neq \emptyset.$$

Obviously,

$$\mathrm{supp}[\psi(\cdot)\varphi(z_0 - \cdot\,)] \subset \mathrm{supp}\varphi(z_0 - \cdot\,)$$

Let $x_0 \in \mathrm{supp}\Lambda \cap \mathrm{supp}\varphi(z_0 - \cdot\,)$ and $y_0 = z_0 - x_0$. We see that $x_0 \in \mathrm{supp}\Lambda$ and $y_0 \in \mathrm{supp}\varphi$. Therefore $\mathrm{supp}(\Lambda * \varphi) \subset \mathrm{supp}\Lambda + \mathrm{supp}\varphi$. As was to be proved. •

Theorem 21 *Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi, \psi \in C^\infty(\mathbb{R})$ be admissible. Then $(\Lambda * \varphi) * \psi = \Lambda * (\varphi * \psi)$.*

Proof. Let us prove the statement in the case $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi, \psi \in C_0^\infty(\mathbb{R})$. We see that

$$\int_{-\infty}^{\infty} \varphi(\,\cdot - y)\psi(y)\,dy = \lim_{\varepsilon \to 0} \sum_{k=-\infty}^{\infty} \varphi(\,\cdot - \varepsilon k)\psi(\varepsilon k)\varepsilon$$

in the space $C_0^\infty(\mathbb{R})$. The sum consists of a finite number of nonzero terms. Therefore,

$$\begin{aligned}
[\Lambda * (\varphi * \psi)](x) &= (\Lambda, \varphi * \psi(x - \cdot\,)) \\
&= \lim_{\varepsilon \to 0} \sum_{k=-\infty}^{\infty} (\Lambda, \varphi(x - \cdot - \varepsilon k))\,\psi(\varepsilon k)\varepsilon \\
&= \lim_{\varepsilon \to 0} \sum_{k=-\infty}^{\infty} \Lambda * \varphi(x - \varepsilon k)\psi(\varepsilon k)\varepsilon \\
&= \int_{-\infty}^{\infty} \Lambda * \varphi(x - y)\psi(y)\,dy \\
&= [(\Lambda * \varphi) * \psi](x)
\end{aligned}$$

for all $x \in R$. •

Theorem 22 *Let $\Lambda \in \mathcal{D}'(\mathbb{R})$. Then*

$$\Lambda = \lim_{\varepsilon \to 0} \Lambda * h_\varepsilon$$

in the space $\mathcal{D}'(\mathbb{R})$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\lim_{\varepsilon\to 0}(\Lambda * h_\varepsilon, \varphi) = \lim_{\varepsilon\to 0}[(\Lambda * h_\varepsilon) * \check{\varphi}](0) = \lim_{\varepsilon\to 0}[\Lambda * (h_\varepsilon * \check{\varphi})](0) = \Lambda * \check{\varphi}\,(0) = (\Lambda, \varphi).$$

As was to be proved. •

Corollary. The space $C_0^\infty(\mathbb{R})$ is dense in the space $\mathcal{D}'(\mathbb{R})$.

Theorem 23 *Let $\Lambda_1, \Lambda_2 \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ be admissible. Then*

$$\Lambda_1 * (\Lambda_2 * \varphi) = \Lambda_2 * (\Lambda_1 * \varphi).$$

Proof. On the one hand, we see that

$$\begin{aligned} \Lambda_1 * [\Lambda_2 * (\varphi * h_\varepsilon)] &= \Lambda_1 * [(\Lambda_2 * \varphi) * h_\varepsilon] \\ &= \Lambda_1 * [h_\varepsilon * (\Lambda_2 * \varphi)] \\ &= (\Lambda_1 * h_\varepsilon) * (\Lambda_2 * \varphi). \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} \Lambda_2 * [\Lambda_1 * (\varphi * h_\varepsilon)] &= \Lambda_2 * [\Lambda_1 * (h_\varepsilon * \varphi)] \\ &= \Lambda_2 * [(\Lambda_1 * h_\varepsilon) * \varphi] \\ &= \Lambda_2 * [\varphi * (\Lambda_1 * h_\varepsilon)] \\ &= (\Lambda_2 * \varphi) * (\Lambda_1 * h_\varepsilon) \\ &= (\Lambda_1 * h\varepsilon) * (\Lambda_2 * \varphi). \end{aligned}$$

Therefore

$$\Lambda_1 * [\Lambda_2 * (\varphi * h_\varepsilon)] = \Lambda_2 * [\Lambda_1 * (\varphi * h_\varepsilon)].$$

We let $\varepsilon \to 0$, and obtain

$$\Lambda_1 * (\Lambda_2 * \varphi) = \Lambda_2 * (\Lambda_1 * \varphi).$$

As was to be proved. •

9. The convolution of distributions

Let $\Lambda_1, \Lambda_2 \in \mathcal{D}'(\mathbb{R})$ be admissible. By definition, this means that $\Lambda_1 \in \mathcal{D}'(\mathbb{R})$ and $\Lambda_2 \in \mathcal{E}'(\mathbb{R})$ or $\Lambda_1 \in \mathcal{E}'(\mathbb{R})$ and $\Lambda_2 \in \mathcal{D}'(\mathbb{R})$. If $\Lambda \in \mathcal{E}'(\mathbb{R})$, then we assume that Λ is extended from $C_0^\infty(\mathbb{R})$ onto $C^\infty(\mathbb{R})$.

The convolution $\Lambda_1 * \Lambda_2$ of the distributions Λ_1 and Λ_2 is called a distribution $\Lambda \in \mathcal{D}'(\mathbb{R})$ such that

$$\Lambda * \varphi = \Lambda_1 * (\Lambda_2 * \varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. We have to prove the existence and the uniqueness of this convolution.

Existence. Define the distribution $\Lambda \in \mathcal{D}'(\mathbb{R})$ by the rule

$$(\Lambda, \varphi) = [\Lambda_1 * (\Lambda_2 * \overset{\vee}{\varphi})](0),$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and $\overset{\vee}{\varphi}(x) = \varphi(-x)$. We claim that

$$\Lambda * \varphi = \Lambda_1 * (\Lambda_2 * \varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. Indeed, let $x \in R$, then

$$\begin{aligned} \Lambda * \varphi(x) &= (\Lambda, \varphi(x - \cdot\,)) \\ &= (\Lambda_1 * \{\Lambda_2 * [\varphi(x + \cdot\,)]\})(0) \\ &= (\Lambda_1 * \{(\Lambda_2 * \varphi)(x + \cdot\,)\})(0) \\ &= ([\Lambda_1 * (\Lambda_2 * \varphi)](x + \cdot\,)\})(0) \\ &= [\Lambda_1 * (\Lambda_2 * \varphi)](x) \end{aligned}$$

Uniqueness. Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ as above, and let $\tilde{\Lambda} \in \mathcal{D}'(\mathbb{R})$ satisfy the identity

$$\tilde{\Lambda} * \varphi = \Lambda_1 * (\Lambda_2 * \varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\Lambda * \varphi = \tilde{\Lambda} * \varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. Therefore

$$(\Lambda, \varphi) = \Lambda * \overset{\vee}{\varphi}(0) = \tilde{\Lambda} * \overset{\vee}{\varphi}(0) = (\tilde{\Lambda}, \varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

Theorem 24 *Let $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{D}'(\mathbb{R})$ be admissible. Then*

$$\begin{aligned} \Lambda_1 * \Lambda_2 &= \Lambda_2 * \Lambda_1, \\ \Lambda_1 * (\Lambda_2 * \Lambda_3) &= (\Lambda_1 * \Lambda_2) * \Lambda_3, \\ \mathrm{supp}(\Lambda_1 * \Lambda_2) &\subset \mathrm{supp}\Lambda_1 + \mathrm{supp}\Lambda_2. \end{aligned}$$

Proof. The first follows from Theorem 23. The second follows from the definition. Let us check the third. Let $z_0 \in \mathrm{supp}(\Lambda_1 * \Lambda_2)$. Then for each $k = 1, 2, \ldots$ there exists $\varphi_k \in C_0^\infty(\mathbb{R})$ such that

$$(\Lambda_1 * \Lambda_2, \varphi_k) = [\Lambda_1 * (\Lambda_2 * \overset{\vee}{\varphi}_k)](0) \neq 0$$

and

$$\mathrm{supp}\varphi_k \subset [z_0 - 1/k, z_0 + 1/k].$$

By Theorem 19, we see that

$$0 \in \mathrm{supp}\Lambda_1 + \mathrm{supp}\Lambda_2 - \mathrm{supp}\varphi_k.$$

Therefore there exist $x_k \in \text{supp}\Lambda_1$, $y_k \in \text{supp}\Lambda_2$ and $| t_k | \leq 1/k$ such that

$$z_0 = x_k + y_k + t_k.$$

Since

$$\lim_{k \to \infty} t_k = 0,$$

and since $\text{supp}\Lambda_1$ or $\text{supp}\Lambda_2$ is compact, there exist $x_0 \in \text{supp}\Lambda_1$ and $y_0 \in \text{supp}\Lambda_2$ such that

$$z_0 = x_0 + y_0.$$

As was to be proved. •

Corollary. If $\Lambda_1, \Lambda_2 \in \mathcal{E}'(\mathbb{R})$, then $\Lambda_1 * \Lambda_2 \in \mathcal{E}'(\mathbb{R})$.

Theorem 25 *Let $\Lambda \in \mathcal{D}'(\mathbb{R})$, then $\Lambda * \delta_0 = \Lambda$.*

Proof. If $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\delta_0 * \varphi(x) = (\delta_0, \varphi(x - \cdot)) = \varphi(x).$$

Therefore

$$(\Lambda * \delta_0) * \varphi = \Lambda * (\delta_0 * \varphi) = \Lambda * \varphi.$$

As was to be proved. •

Remark. Let $\Lambda_1, \Lambda_2 \in \mathcal{E}'(\mathbb{R})$. It can be proved that

$$\begin{aligned} \min \text{supp}(\Lambda_1 * \Lambda_2) &= \min \text{supp}\Lambda_1 + \min \text{supp}\Lambda_2, \\ \max \text{supp}(\Lambda_1 * \Lambda_2) &= \max \text{supp}\Lambda_1 + \max \text{supp}\Lambda_2. \end{aligned}$$

Theorem 26 *Let $\Lambda_1, \Lambda_2 \in \mathcal{D}'(\mathbb{R})$ be admissible. Then*

$$(\Lambda_1 * \Lambda_2)' = \Lambda_1' * \Lambda_2 = \Lambda_1 * \Lambda_2'.$$

Proof. Let $\Lambda \in \mathcal{D}'(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ be admissible. We claim that

$$(\Lambda * \varphi)' = \Lambda' * \varphi.$$

Indeed, we have

$$(\Lambda * \varphi)'(x) = (\Lambda * \varphi')(x) = (\Lambda, \varphi'(x - \cdot\,) = -(\Lambda, -\varphi'(x - \cdot\,) = (\Lambda', \varphi(x - \cdot\,) = (\Lambda' * \varphi)(x).$$

Let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}
((\Lambda_1 * \Lambda_2)', \varphi) &= -(\Lambda_1 * \Lambda_2, \varphi') \\
&= -((\Lambda_1 * \Lambda_2) * (\varphi')^\vee)(0) \\
&= \left((\Lambda_1 * \Lambda_2) * \overset{\vee}{\varphi}{}'\right)(0) \\
&= \left(\Lambda_1 * (\Lambda_2 * \overset{\vee}{\varphi}{}'\right)(0) \\
&= \left(\Lambda_1 * (\Lambda_2' * \overset{\vee}{\varphi}\right)(0) \\
&= \left((\Lambda_1 * \Lambda_2') * \overset{\vee}{\varphi}\right)(0) \\
&= (\Lambda_1 * \Lambda_2', \varphi)
\end{aligned}$$

As was to be proved. $\bullet$

Theorem 27 *Let $f, g \in L_1(\mathbb{R})$ be admissible. Then $\Lambda_{f*g} = \Lambda_f * \Lambda_g$.*

Proof. We recall that

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

Let $\varphi \in C_0^\infty(\mathbb{R})$. Then

$$\Lambda_g * \varphi(x) = (\Lambda_g, \varphi(x - \cdot\,)) = \int_{-\infty}^{\infty} g(y)\varphi(x-y)\,dy.$$

Therefore

$$\begin{aligned}
(\Lambda_f * (\Lambda_g * \varphi))(z) &= \int_{-\infty}^{\infty} f(x) \left\{ \int_{-\infty}^{\infty} g(y)\varphi(z-y-x)\,dy \right\} dx \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t-y)g(y)\,dy \right\} \varphi(z-t)\,dt \\
&= \int_{-\infty}^{\infty} f * g(t)\varphi(z-t)\,dt \\
&= (\Lambda_{f*g}, \varphi(z - \cdot\,)) \\
&= \Lambda_{f*g} * \varphi(z)
\end{aligned}$$

for all $z \in R$. Therefore

$$\Lambda_{f*g} * \varphi = \Lambda_f * (\Lambda_g * \varphi).$$

As was to be proved. $\bullet$

10. Differential operators

Consider the differential operator

$$\mathcal{L} = \sum_{k=0}^{l} a_k \frac{d^k}{dx^k}$$

and the dual differential operator

$$\overset{\vee}{\mathcal{L}} = \sum_{k=0}^{l} (-1)^k a_k \frac{d^k}{dx^k}.$$

We assume that $a_l = 1$.

We see that

$$(\mathcal{L}\Lambda, \varphi) = (\Lambda, \overset{\vee}{\mathcal{L}}\, \varphi)$$

for all $\Lambda \in \mathcal{D}'(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Besides, the linear operators

$$\mathcal{L} : \ \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

and

$$\overset{\vee}{\mathcal{L}} : \ \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

are continuous.

Let Z satisfy the differential equation

$$\mathcal{L} Z = 0$$

with initial conditions

$$\left.\begin{array}{ll} Z^{(0)}(0) & = 0 \\ \cdots & \\ Z^{(l-2)}(0) & = 0 \end{array}\right\} \quad \text{if} \quad l \geq 2$$
$$Z^{(l-1)}(0) = 1$$

Theorem 28 *There exists a linear continuous operator*

$$I_{\mathcal{L}} : \ \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

such that $\mathcal{L} I_{\mathcal{L}}(\Lambda) = \Lambda$ *for all* $\Lambda \in \mathcal{D}'(\Omega)$.

Proof. By Corollary to Theorem 17, there exists a linear continuous operator

$$I : \ \mathcal{D}'(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

such that

$$\frac{d}{dx} I(\Lambda) = \Lambda$$

for all $\Lambda \in \mathcal{D}'(\Omega)$.

Let

$$\Phi = \begin{pmatrix} Z^{(0)} & Z^{(1)} & \dots & Z^{(l-1)} \\ Z^{(1)} & Z^{(2)} & \dots & Z^{(l)} \\ \dots & \dots & \dots & \dots \\ Z^{(l-1)} & Z^{(l)} & \dots & Z^{(2l-2)} \end{pmatrix}.$$

It is not hard to check that $\det \Phi(x) \neq 0$ for all $x \in \mathbb{R}$.

We claim that

$$I_{\mathcal{L}}(\Lambda) = \sum_{k=0}^{l-1} Z^{(k)} I(\varphi_k \Lambda),$$

where $\varphi_0, \dots, \varphi_{l-1} \in C^\infty(\Omega)$ are elements of the last column of the matrix Φ^{-1}.

Put

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{l-1} \end{pmatrix} \qquad F = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Lambda \end{pmatrix} \qquad U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{l-2} \\ u_{l-1} \end{pmatrix},$$

where $\Lambda \in \mathcal{D}'(\Omega)$ and $u_0, \dots, u_{l-1} \in \mathcal{D}'(\Omega)$. Note that

$$\Phi' = A\Phi$$

Consider the differential equation

$$\mathcal{L}u = \Lambda \tag{1}$$

and the system of differential equations

$$U' = AU + F. \tag{2}$$

We see that (1) and (2) are equivalent in the following sense. If u is a solution of (1), then U with $u_k = u^{(k)}$ is a solution of (2). If U is a solution of (2), then $u = u_0$ is a solution of (1). Let

$$U = \Phi V.$$

Since

$$U' = \Phi' V + \Phi V' = A\Phi V + \Phi V' = AU + \Phi V',$$

we see that (2) is equivalent to

$$V' = \Phi^{-1} F. \tag{3}$$

From (3) we have

$$\begin{aligned} v_0' &= \varphi_0 \Lambda, \\ &\dots \\ v_{l-1}' &= \varphi_{l-1} \Lambda. \end{aligned}$$

Therefore

$$\begin{array}{lcl} v_0 & = & I(\varphi_0\Lambda) + c_0, \\ & \cdots & \\ v_{l-1} & = & I(\varphi_{l-1}\Lambda) + c_{l-1} \end{array}$$

with some constants $c_0, \ldots, c_{l-1}$. Since $U = \Phi V$, we see that $u = u_0$ has the representation

$$u = \sum_{k=0}^{l-1} Z^{(k)} I(\varphi_k \Lambda) + \sum_{k=0}^{l-1} c_k Z^{(k)}.$$

We rewrite the last formula in the form

$$u = \sum_{k=0}^{l-1} c_k Z^{(k)} + I_{\mathcal{L}}(\Lambda).$$

and obtain

$$\mathcal{L} I_{\mathcal{L}}(\Lambda) = \Lambda.$$

As was to be proved. •

A distribution $E \in \mathcal{D}'(R)$ such that

$$\mathcal{L}E = \delta_0$$

is called a fundamental solution of $\mathcal{L}$.

Theorem 29 *The function $E = Z\Theta$ is the fundamental solution of $\mathcal{L}$.*

Proof. By Theorem 14, we have

$$\begin{array}{lcl} E^{(0)} & = & Z^{(0)}\Theta \\ E^{(1)} & = & Z^{(1)}\Theta \\ & \cdots & \\ E^{(l-1)} & = & Z^{(l-1)}\Theta \\ E^{(l)} & = & Z^{(l)}\Theta + \delta_0 \end{array}$$

Therefore

$$\mathcal{L}E = (\mathcal{L}Z)\Theta + \delta_0 = \delta_0.$$

As was to be proved. •

Let us put

$$E = Z\Theta.$$

We see that

$$\overset{\vee}{E}(x) = E(-x)$$

is a fundamental solution of $\overset{\vee}{\mathcal{L}}$.

Theorem 30 *Let $\Lambda \in \mathcal{E}'(\mathbb{R})$. Then*

$$\mathcal{L}(E * \Lambda) = \Lambda.$$

Proof. Indeed, we see that

$$\mathcal{L}(E * \Lambda) = (\mathcal{L}E) * \Lambda = \delta * \Lambda = \Lambda.$$

As was to be proved. •

Remark. Let

$$\mathcal{L}(\lambda) = \lambda^l + a_{l-1}\lambda^{l-1} + \cdots + a_0 = (\lambda - \lambda_1)^{q_1} \ldots (\lambda - \lambda_m)^{q_m}$$

where q_i is the multiplicity of the root λ_i. The function Z is representable in the form

$$Z(x) = \sum_{i=1}^{m} \left(\sum_{k=1}^{q_i} C_{ik} \frac{x^k}{k!} \right) e^{\lambda_i x},$$

where

$$C_{ik} = \frac{1}{(q_i - k)!} \left[\frac{(\lambda - \lambda_i)^{q_i}}{\mathcal{L}(\lambda)} \right]^{(q_i - k)} \Bigg|_{\lambda = \lambda_i}.$$

11. Sobolev spaces

Let $l \geq 0$ be an integer number. The space of all distributions $U \in \mathcal{D}'(\Omega)$ such that $U^{(k)} \in L_2(\Omega)$ for all $k = 0, 1, \ldots, l$ is denoted by $W_2^l(\Omega)$ and is called the Sobolev space. The Sobolev space is a Hilbert space with the scalar product

$$(U, V)_{W_2^l(\Omega)} = \sum_{k=0}^{l} (U^{(k)}, V^{(k)})_{L_2(\Omega)}$$

and the norm

$$\|U\|^2_{W_2^l(\Omega)} = \sum_{k=0}^{l} \|U^{(k)}\|^2_{L_2(\Omega)}.$$

Let $C^m[a, b]$ be the Banach space of C^m-functions with the norm

$$\|U\|_{C^k[a,b]} = \sum_{k=0}^{m} \|U^{(k)}\|_{C[a,b]}.$$

Theorem 31 *Let $l \geq 1$. Then $W_2^l(\Omega) \subset C^{l-1}[a,b]$ is an embedding.*

Proof. Let $\alpha \in \Omega$ and $U \in W_2^l(\Omega)$. We see that

$$U^{(k)}(x) = U^{(k)}(\alpha) + \int_\alpha^x U^{(k+1)}(t)\,dt$$

for all $k = 0, 1, \ldots, l-1$. This means that $U \in C^{l-1}[a,b]$.

From

$$U^{(k)}(\alpha) = U^{(k)}(x) - \int_\alpha^x U^{(k+1)}(t)\,dt.$$

we obtain the inequality

$$(b-a)^{1/2} \mid U^{(k)}(\alpha) \mid \leq \|U^{(k)}\|_{L_2(\Omega)} + (b-a)^{1/2}\|U^{(k+1)}\|_{L_2(\Omega)}.$$

Therefore,

$$\|U^{(k)}\|_{C[a,b]} \leq (b-a)^{-1/2}\|U^{(k)}\|_{L_2(\Omega)} + \|U^{(k+1)}\|_{L_2(\Omega)}.$$

This means that

$$\|U\|_{C^{l-1}[a,b]} \leq C\|U\|_{W_2^l(\Omega)}$$

with some constant C. Hence $W_2^l(\Omega) \subset C^{l-1}[a,b]$ is an embedding. •

Theorem 32 *Let $U \in W_2^l(\Omega)$ and $\alpha \in \Omega$. Then*

$$U(x) = \sum_{k=0}^{l-1} U^{(k)}(\alpha)\frac{(x-\alpha)^k}{k!} + \int_\alpha^x \frac{(x-y)^{l-1}}{(l-1)!}U^{(l)}(y)\,dy.$$

Proof. We see that

$$\int_\alpha^x \frac{(x-y)^k}{k!}f(y)\,dy = \int_\alpha^x \left(\int_\alpha^y \frac{(y-t)^{k-1}}{(k-1)!}f(t)\,dt\right)\,dy.$$

Indeed,

$$\begin{aligned}
\int_\alpha^x \left(\int_\alpha^y \frac{(y-t)^{k-1}}{(k-1)!}f(t)\,dt\right)\,dy &= \int_\alpha^x \left(\int_\alpha^x \frac{(y-t)_+^{k-1}}{(k-1)!}f(t)\,dt\right)\,dy \\
&= \int_\alpha^x \left(\int_\alpha^x \frac{(y-t)_+^{k-1}}{(k-1)!}\,dy\right)f(t)\,dt \\
&= \int_\alpha^x \left(\int_t^x \frac{(y-t)^{k-1}}{(k-1)!}\,dy\right)f(t)\,dt \\
&= \int_\alpha^x \frac{(x-t)^k}{k!}f(t)\,dt.
\end{aligned}$$

Since

$$U^{(k)}(x) = U^{(k)}(\alpha) + \int_{\alpha}^{x} U^{(k+1)}(t)\, dt,$$

where $k = 0, 1, \ldots, l-1$, we have

$$U(x) = U(\alpha) + \int_{\alpha}^{x} U'(y)\, dy = U(\alpha) + U'(\alpha)\frac{(x-\alpha)}{1!} + \int_{\alpha}^{x} \frac{x-y}{1!} U''(y)\, dy = \cdots$$

As was to be proved. •

Theorem 33 *The space $C^l[a,b]$ is dense in the space $W_2^l(\Omega)$.*

Proof. Let $U \in W_2^l(\Omega)$ and $\alpha \in \Omega$. Then

$$U(x) = \sum_{k=0}^{l-1} U^{(k)}(\alpha)\frac{(x-\alpha)^k}{k!} + \int_{\alpha}^{x} \frac{(x-y)^{l-1}}{(l-1)!} U^{(l)}(y)\, dy.$$

Since $U^{(l)} \in L_2(\Omega)$, given $\varepsilon > 0$, there exists $f \in C[a,b]$ such that

$$\|U^{(l)} - f\|_{L_2(\Omega)} \leq \varepsilon.$$

Let

$$F(x) = \sum_{k=0}^{l-1} U^{(k)}(\alpha)\frac{(x-\alpha)^k}{k!} + \int_{\alpha}^{x} \frac{(x-y)^{l-1}}{(l-1)!} f(y)\, dy.$$

We see that $F \in C^l[a,b]$ and

$$\|U - F\|_{W_2^l(\Omega)} \leq C\varepsilon$$

with some constant $C > 0$. •

Let us define the space

$$\overset{0}{W}{}_2^l(\Omega) = \mathrm{Cl} C_0^{\infty}(\Omega)$$

where Cl is the closure in the space $W_2^l(\Omega)$.

Theorem 34 *Let $U \in W_2^l(\Omega)$. Then $U \in \overset{0}{W}{}_2^l(\Omega)$ if and only if*

$$U^{(k)}(a) = U^{(k)}(b) = 0$$

for all $0 \leq k \leq l-1$.

Proof. Sufficiency. Let $U^{(k)}(a)=U^{(k)}(b)=0$ for all $0\le k\le l-1$. Define the function

$$V_\varepsilon(x)=\int\limits_{a+2\varepsilon}^{b-2\varepsilon} h_\varepsilon(x-y)\,dy.$$

We claim that

$$\lim_{\varepsilon\to 0}\|U-V_\varepsilon U\|_{W_2^l(\Omega)}=0.$$

We see that

$$(V_\varepsilon U)^{(k)}=V_\varepsilon U^{(k)}+\sum_{p=1}^{k} C_k^p V_\varepsilon^{(p)} U^{(k-p)}$$

and

$$\lim_{\varepsilon\to 0}\|U^{(k)}-V_\varepsilon U^{(k)}\|_{L_2(\Omega)}=0$$

for all $0\le k\le l$. Let us prove that

$$\lim_{\varepsilon\to 0}\|V_\varepsilon^{(p)} U^{(k-p)}\|_{L_2(\Omega)}=0$$

for all $1\le p\le k\le l$.

Since

$$V_\varepsilon^{(p)}(x)=\int\limits_{a+2\varepsilon}^{b-2\varepsilon} h_\varepsilon^{(p)}(x-y)\,dy,$$

we have

$$\|V_\varepsilon^{(p)}\|_{C[a,b]}\le\frac{C}{\varepsilon^p}$$

where $C>0$ does not depend on $\varepsilon>0$ and $0\le p\le l$. Since

$$U^{(k)}(a)=U^{(k)}(b)=0,$$

where $0\le k\le l-1$, we have

$$U^{(k-p)}(x)=\int\limits_a^x\frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!}U^{(l)}(y)\,dy$$

and

$$U^{(k-p)}(x)=\int\limits_b^x\frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!}U^{(l)}(y)\,dy,$$

where $1\le p\le k\le l-1$. Therefore,

$$\begin{aligned}
\|V_\varepsilon^{(p)}U^{(k-p)}\|^2_{L_2(\Omega)} &= \int\limits_a^b \mid V_\varepsilon^{(p)}(x)U^{(k-p)}(x)\mid^2\ dx\\
&= \int\limits_a^{a+\varepsilon}\mid V_\varepsilon^{(p)}(x)\mid^2\left|\int\limits_a^x\frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!}U^{(l)}(y)\,dy\right|^2\ dx
\end{aligned}$$

$$
\begin{aligned}
& + \int_{b-\varepsilon}^{b} \mid V_\varepsilon^{(p)}(x) \mid^2 \left| \int_b^x \frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!} U^{(l)}(y)\,dy \right|^2 dx \\
\leq \; & \frac{C}{\varepsilon^{2p}} \left\{ \int_a^{a+\varepsilon} \left| \int_a^x \frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!} U^{(l)}(y)\,dy \right|^2 dx \right. \\
& \left. + \int_{b-\varepsilon}^{b} \left| \int_b^x \frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!} U^{(l)}(y)\,dy \right|^2 dx \right\} \\
\leq \; & \frac{C}{\varepsilon^{2p}} \left\{ \int_a^{a+\varepsilon} \mid U^{(l)}(y) \mid^2 \; dy \int_a^{a+\varepsilon} \int_a^x \left[\frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!} \right]^2 dy dx \right. \\
& \left. + \int_{b-\varepsilon}^{b} \mid U^{(l)}(y) \mid^2 \; dy \int_{b-\varepsilon}^{b} \int_x^b \left[\frac{(x-y)^{l-k+p-1}}{(l-k+p-1)!} \right]^2 dy dx \right\} \\
\leq \; & \frac{C}{\varepsilon^{2p}} \varepsilon^{2(m-k+p-1)+2} \left\{ \int_a^{a+\varepsilon} \mid U^{(l)}(y) \mid^2 \; dy + \int_{b-\varepsilon}^{b} \mid U^{(l)}(y) \mid^2 \; dy \right\} \\
= \; & C\varepsilon^{2(m-k)} \left\{ \int_a^{a+\varepsilon} \mid U^{(l)}(y) \mid^2 \; dy + \int_{b-\varepsilon}^{b} \mid U^{(l)}(y) \mid^2 \; dy \right\}
\end{aligned}
$$

This means that

$$\lim_{\varepsilon \to 0} \| V_\varepsilon^{(p)} U^{(k-p)} \|_{L_2(\Omega)} = 0$$

for all $1 \leq p \leq k \leq l$. Thus

$$\lim_{\varepsilon \to 0} \| U - V_\varepsilon U \|_{W_2^l(\Omega)} = 0.$$

Note that $\text{supp}(V_\varepsilon U) \subset [a+\varepsilon, b-\varepsilon]$.

Let $u \in L_2(\Omega)$. We assume that u is extended by zero outside of Ω. Let us check the inequality

$$\| u * h_\varepsilon \|_{L_2(\Omega)} \leq \| u \|_{L_2(\Omega)}.$$

Indeed,

$$
\begin{aligned}
\| u * h_\varepsilon \|_{L_2(\Omega)} \; & = \; \int_a^b \left\{ \int_{-\varepsilon}^{\varepsilon} [u(x-y) h_\varepsilon(y)^{1/2}] h_\varepsilon(y)^{1/2}\,dy \right\} dx \\
& \leq \; \int_a^b \left\{ \int_{-\varepsilon}^{\varepsilon} \mid u(x-y) \mid^2 h_\varepsilon(y)\,dy \int_{-\varepsilon}^{\varepsilon} h_\varepsilon(y)\,dy \right\} dx \\
& \leq \; \int_{-\varepsilon}^{\varepsilon} h_\varepsilon(y) \int_{-\infty}^{\infty} \mid u(x-y) \mid^2 h_\varepsilon(y)\,dx dy \\
& = \; \| u \|^2_{L_2(\Omega)}.
\end{aligned}
$$

Thus for any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that

$$\|U - V_{\varepsilon_1} U\|_{W_2^l(\Omega)} \leq \frac{\varepsilon}{2}.$$

Then there exists $\varepsilon_2 > 0$ such that

$$\|V_{\varepsilon_1} U - (V_{\varepsilon_1} U) * h_{\varepsilon_2}\|_{W_2^l(\Omega)} \leq \frac{\varepsilon}{2}.$$

We see that

$$\|U - (V_{\varepsilon_1} U) * h_{\varepsilon_2}\|_{W_2^l(\Omega)} \leq \varepsilon$$

and

$$(V_{\varepsilon_1} U) * h_{\varepsilon_2} \in C_0^\infty(\Omega).$$

This proves the inclusion $U \in \overset{0}{W}{}_2^l(\Omega)$.

Necessity. Let $U \in \overset{0}{W}{}_2^l(\Omega)$. Then there exists a sequence $U_1, U_2, \ldots \in C_0^\infty(\Omega)$ such that

$$\lim_{k\to\infty} U_k = U$$

in the space $W_2^l(\Omega)$. Since

$$U_k^{(p)}(a) = U_k^{(p)}(b) = 0$$

for all $0 \leq p \leq l-1$, we have

$$U^{(p)}(a) = U^{(p)}(b) = 0$$

for all $0 \leq p \leq l-1$. •

Consider the cone

$$\overset{+}{C}{}_0^\infty(\Omega) = \{\varphi \in C_0^\infty(\Omega) : \ \varphi \geq 0\}$$

of positive functions in the space $C_0^\infty(\Omega)$ and the cone

$$\overset{+}{W}{}_2^l(\Omega) = \{U \in \overset{0}{W}{}_2^l(\Omega) : \ U \geq 0\}$$

of positive functions in the space $\overset{0}{W}{}_2^l(\Omega)$.

Corollary 1. Let $U \in \overset{0}{W}{}_2^l(\Omega)$. Then given $\varepsilon > 0$, there exists $V \in \overset{+}{C}{}_0^\infty(\Omega)$ such that

$$\|U - VU\|_{W_2^l(\Omega)} \leq \varepsilon.$$

Corollary 2. We have $\overset{+}{W}{}_2^l(\Omega) = \mathrm{Cl}\ \overset{+}{C}{}_0^\infty(\Omega)$ where Cl is the closure in the space $W_2^l(\Omega)$.

Corollary 3. Let $U \in W_2^l(\Omega)$ and $U^{(k)}(a) = U^{(k)}(b) = 0$ for all $0 \leq k \leq l-1$. If U is extended by zero outside of Ω, then $U \in \overset{0}{W}{}_2^l(\mathbb{R})$.

12. The Green function

Consider the differential operators

$$\mathcal{L} = \sum_{k=0}^{l} a_k \frac{d^k}{dx^k}$$

and

$$\overset{\vee}{\mathcal{L}} = \sum_{k=0}^{l} (-1)^k a_k \frac{d^k}{dx^k}.$$

Let $f \in L_2(\Omega)$. We assume that f is extended by zero outside of Ω. Hence $f \in L_2(\mathbb{R})$ and $\mathrm{supp} f \subset [a,b]$. In particular, $f \in \mathcal{E}'(R)$. By Theorem 30 and Theorem 27, the differential equation

$$\mathcal{L}u = f$$

has a solution $u = E * f$. We see that

$$u(x) = \int_a^b E(x-y)f(y)\,dy = \int_a^x Z(x-y)f(y)\,dy.$$

Define the operator

$$\overset{\vee}{E} *: \ L_2(\Omega) \longrightarrow W_2^l(\Omega)$$

by the rule

$$\overset{\vee}{E} *: \ u \longmapsto (\overset{\vee}{E} * u)\,|_{\Omega}\ .$$

We see that

$$\overset{\vee}{E} * u(x) = \int_x^b Z(y-x)u(y)\,dy.$$

Theorem 35 *The operator*

$$\overset{\vee}{E} *: \ L_2(\Omega) \longrightarrow W_2^l(\Omega)$$

is continuous.

Proof. Let $u \in L_2(\Omega)$. Since

$$\overset{\vee}{E}{}^{(k)} = (-1)^k (Z^{(k)}\Theta)^{\vee}$$

for $0 \le k \le l-1$ and

$$\overset{\vee}{E}{}^{(l)} = (-1)^l (Z^{(l)}\Theta)^{\vee} + (-1)^l \delta_0,$$

we have

$$(\overset{\vee}{E} * u)^{(k)}(x) = (\overset{\vee}{E}{}^{(k)} * u)(x) = (-1)^k \int_x^b Z^{(k)}(y-x)u(y)\,dy$$

for $0 \le k \le l-1$ and

$$(\overset{\vee}{E} * u)^{(l)}(x) = (\overset{\vee}{E}{}^{(l)} * u)(x) = (-1)^l \int_x^b Z^{(l)}(y-x)u(y)\,dy + (-1)^l u(x).$$

Therefore

$$\| \overset{\vee}{E} * u\|_{W_2^l(\Omega)} \le C\|u\|_{L^2(\Omega)}$$

where $C > 0$ does not depend on $u \in L_2(\Omega)$. •

Define the space

$$P_{\mathcal{L}}(\Omega) = \{u \in L_2(\Omega) : \ \mathcal{L}u = 0\}.$$

Obviously

$$\overset{\vee}{\mathcal{L}}\colon \ W_2^l(\Omega) \longrightarrow L_2(\Omega)$$

is a linear continuous operator.

Theorem 36 *The operators*

$$\begin{array}{llll} \overset{\vee}{E} * : & P_{\mathcal{L}}(\Omega)^{\perp} & \longrightarrow & \overset{0}{W}{}_2^l(\Omega) \\ \overset{\vee}{\mathcal{L}}\colon & \overset{0}{W}{}_2^1(\Omega) & \longrightarrow & P_{\mathcal{L}}(\Omega)^{\perp} \end{array}$$

are inverse continuous isomorphisms.

Proof. **1.** Let $u \in P_{\mathcal{L}}(\Omega)^{\perp}$ and $U = \overset{\vee}{E} * u$. We note that $Z^{(k)}(\cdot - x) \in P_{\mathcal{L}}(\Omega)$ for all $k = 0, 1, \ldots, l-1$. Since

$$(\overset{\vee}{E} * u)^{(k)}(x) = (-1)^k \int_x^b Z^{(k)}(y-x)u(y)\,dy$$

for all $k = 0, 1, \ldots, l-1$, we have

$$U(a) = (\overset{\vee}{E} * u)^{(k)}(a) = (-1)^k \int_a^b Z^{(k)}(y-a)u(y)\,dy = 0$$

and

$$U(b) = (\overset{\vee}{E} * u)^{(k)}(b) = (-1)^k \int_b^b Z^{(k)}(y-b)u(y)\,dy = 0$$

all $k = 0, 1, \ldots, l-1$. By Theorem 34, we conclude that $U \in \overset{0}{W}{}_2^l(\Omega)$.

2. Let $U \in \overset{0}{W}{}_2^l(\Omega)$ and $v \in P_{\mathcal{L}}(\Omega)$. Let $\varphi_1, \varphi_2, \ldots \in C_0^{\infty}(\Omega)$ be such that

$$\lim_{k\to\infty} \varphi_k = U$$

in the space $W_2^l(\Omega)$. Then

$$(\overset{\vee}{\mathcal{L}}\, U, v)_{L_2(\Omega)} = \lim_{k\to\infty} (\overset{\vee}{\mathcal{L}}\, \varphi_k, v)_{L_2(\Omega)} = \lim_{k\to\infty} (\varphi_k, \mathcal{L}v)_{L_2(\Omega)} = 0$$

Therefore $\overset{\vee}{\mathcal{L}}\, U \in P_{\mathcal{L}}(\Omega)^{\perp}$.

3. By Theorem 30, we have

$$\overset{\vee}{\mathcal{L}}\, (\overset{\vee}{E} *u) = u$$

for all $u \in L_2(\Omega)$. Let $U \in \overset{0}{W}{}_2^l(\Omega)$ and $\tilde{U} = \overset{\vee}{E} *(\overset{\vee}{\mathcal{L}}\, U)$. Obviously $\overset{\vee}{\mathcal{L}}\, \tilde{U} = \overset{\vee}{\mathcal{L}}\, U$ Since $\overset{\vee}{\mathcal{L}}\, (\tilde{U} - U) = 0$, and since $\tilde{U} - U \in \overset{0}{W}{}_2^l(\Omega)$, we have $\tilde{U} = U$. Therefore

$$\overset{\vee}{E} *(\overset{\vee}{\mathcal{L}}\, U) = U$$

for all $U \in \overset{0}{W}{}_2^l(\Omega)$. •

Theorem 37 *The space* $\overset{0}{W}{}_2^l(\Omega)$ *is a Hilbert space with the scalar product*

$$(U, V)_{\mathcal{L}} = (\overset{\vee}{\mathcal{L}}\, U, \overset{\vee}{\mathcal{L}}\, V)_{L_2(\Omega)}$$

and the norm

$$\|U\|_{\mathcal{L}} = \|\, \overset{\vee}{\mathcal{L}}\, U\|_{L_2(\Omega)}.$$

Proof. We claim that the norms $\|\cdot\|_{\mathcal{L}}$ and $\|\cdot\|_{W_2^l(\Omega)}$ are equivalent in the space $\overset{0}{W}{}_2^l(\Omega)$. This means that

$$C_0\|U\|_{W_2^l(\Omega)} \le \|U\|_{\mathcal{L}} \le C_1\|U\|_{W_2^l(\Omega)}$$

for some constants $C_0, C_1 > 0$. Let $U \in \overset{0}{W}{}_2^l(\Omega)$ and $u = \overset{\vee}{\mathcal{L}}\, U$.

On the one hand,

$$(\overset{\vee}{E} *u)^{(k)}(x) = (-1)^k \int_x^b Z^{(k)}(y - x)u(y)\, dy$$

for $0 \le k \le l - 1$ and

$$(\overset{\vee}{E} *u)^{(l)}(x) = (-1)^l \int_x^b Z^{(l)}(y - x)u(y)\, dy + (-1)^l u(x).$$

Hence

$$U^{(k)}(x) = (-1)^k \int_x^b Z^{(k)}(y - x)\, \overset{\vee}{\mathcal{L}}\, U(y)\, dy$$

for $0 \le k \le l-1$ and

$$U^{(l)}(x) = (-1)^l \int\limits_x^b Z^{(l)}(y-x) \overset{\vee}{\mathcal{L}} U(y)\, dy + (-1)^l \overset{\vee}{\mathcal{L}} U(x).$$

Therefore

$$\|U\|_{W_2^l(\Omega)} \le C\|U\|_{\mathcal{L}}$$

with a constant $C > 0$.

On the other hand,

$$\begin{aligned}
\|U\|_{\mathcal{L}}^2 &= (\overset{\vee}{\mathcal{L}} U, \overset{\vee}{\mathcal{L}} U)_{L_2(\Omega)} \\
&= \sum_{i=0}^{l}\sum_{j=0}^{l}(-1)^{i+j} a_i a_j (U^{(i)}, U^{(j)})_{L_2(\Omega)} \\
&\le \sum_{i=0}^{l}\sum_{j=0}^{l} \mid a_i \mid\mid a_j \mid \|U^{(i)}\|_{L_2(\Omega)} \|U^{(j)})\|_{L_2(\Omega)} \\
&= \left(\sum_{i=0}^{l} \mid a_i \mid \|U^{(i)}\|_{L_2(\Omega)}\right)^2 \\
&\le C\|U\|^2_{W_2^l(\Omega)}
\end{aligned}$$

with a constant $C > 0$. $\bullet$

The function $G(x,y)$ is called the Green function of the differential operator $\mathcal{L} \overset{\vee}{\mathcal{L}}$ if

$$\mathcal{L} \overset{\vee}{\mathcal{L}} G(\,\cdot\,, y) = \delta_y$$

and

$$G(\,\cdot\,, y) \in \overset{0}{W_2^l}(\Omega)$$

for any $y \in \Omega$.

Let $\mathcal{E}$ be a fundamental solution of the differential operator $\mathcal{L} \overset{\vee}{\mathcal{L}}$, and let the function $R(x,y)$ satisfy the differential equation

$$\mathcal{L}_x \overset{\vee}{\mathcal{L}}_x R(x,y) = 0$$

with boundary conditions

$$\begin{aligned}
R_x^{(k)}(a,y) &= \mathcal{E}^{(k)}(a-y), \\
R_x^{(k)}(b,y) &= \mathcal{E}^{(k)}(b-y),
\end{aligned}$$

where $k = 0, 1, \ldots, l-1$. Then obviously

$$G(x,y) = \mathcal{E}(x-y) - R(x,y)$$

the Green function of the differential operator $\mathcal{L}\overset{\vee}{\mathcal{L}}$.

Define the function

$$\gamma_y = \mathcal{P}_{P_L(\Omega)^\perp}\left\{E(\,\cdot\, - y)\,|_\Omega\right\},$$

where

$$\mathcal{P}_{P_L(\Omega)^\perp} :\ L_2(\Omega) \longrightarrow P_L(\Omega)^\perp$$

is the projection.

Theorem 38 *The Green function of the operator $\mathcal{L}\overset{\vee}{\mathcal{L}}$ has the representation*

$$G(x,y) = (\gamma_x, \gamma_y)_{L_2(\Omega)}.$$

Proof. We see that

$$(\gamma_x, \gamma_y)_{L_2(\Omega)} = (\overset{\vee}{E} * \gamma_y)(x).$$

Since $\gamma_y \in P_L(\Omega)$, by Theorem 36, we have

$$G(\,\cdot\,, y) \in \overset{0}{W}{}_2^l(\Omega).$$

Then

$$\mathcal{L}\overset{\vee}{\mathcal{L}}\,(\overset{\vee}{E} ** \gamma_y) = \mathcal{L}\gamma_y = \delta_y.$$

This proves the representation. •

For example, calculate the Green function $G(x,y)$ for the differential operator

$$\mathcal{L} = \frac{d^l}{dx^l}$$

in the interval $\Omega = (-1, 1)$.

Theorem 39 *The Green function of the differential operator*

$$(-1)^l \frac{d^{2l}}{dx^{2l}}$$

has the representation

$$G(x,y) = \frac{(-1)^l}{2(2l-1)!} \mid x - y \mid^{2l-1} - R(x,y)$$

with

$$R(x,y) = \frac{(-1)^l}{2(2l-1)!}(1 - xy)\sum_{k=0}^{l-1} C_k[(x^2-1)(y^2-1)]^k (x-y)^{2(l-1-k)},$$

where

$$C_k = \frac{((1+\lambda)^{-1/2})^{(k)}}{k!}(0) = \frac{(-1)^k}{2^{k-1}} C_{2k-1}^k.$$

Proof. Let $y \in \Omega$. Since

$$(-1)^l \frac{d^{2l}}{dx^{2l}} G(x,y) = \delta_y,$$

we have only check the boundary conditions. Let $x \neq y$, then

$$\frac{x^2y^2 - 2xy + 1}{\mid x - y \mid^2} = 1 + \frac{(x^2-1)(y^2-1)}{\mid x - y \mid^2}.$$

Therefore

$$G(x,y) = \frac{(-1)^l}{2(2l-1)!} \mid x - y \mid^{2l-1} g\left(\frac{(x^2-1)(y^2-1)}{\mid x - y \mid^2}\right)$$

where

$$g(\lambda) = (1+\lambda)^{1/2}\Big\{(1+\lambda)^{-1/2} - \sum_{k=0}^{l-1} C_k \lambda^k\Big\}.$$

Since

$$g(0) = g'(0) = \ldots = g^{(l-1)}(0) = 0,$$

we have

$$\frac{d^k}{dx^k} G(-1,y) = \frac{d^k}{dx^k} G(1,y) = 0$$

for all $k = 0, 1, \ldots, l-1$. •

Corollary. We have $G(x,y) > 0$ for all $x, y \in \Omega$.
Proof. Obviously $G(x,x) > 0$ for all $x \in \Omega$. Let $x \neq y$. By Theorem 39, we have

$$G(x,y) = \frac{(-1)^l}{2(2l-1)!} \mid x - y \mid^{2l-1} (1+\lambda)^{1/2} f(\lambda)$$

where

$$\lambda = \frac{(x^2-1)(y^2-1)}{\mid x - y \mid^2}$$

and

$$f(\lambda) = (1+\lambda)^{-1/2} - \sum_{k=0}^{l-1} C_k \lambda^k.$$

We note that

$$f(0) = f'(0) = \ldots = f^{(l-1)}(0) = 0$$

and

$$f^{(l)}(\lambda) = (-1)^l \frac{1}{2}\frac{3}{2} \cdots \frac{2l-1}{2} (1+\lambda)^{-1/2-l}.$$

Therefore

$$\text{sign} f(\lambda) = (-1)^l$$

for all $\lambda > 0$. •

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