Continued Fractions of Cubic Irrationalities

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Abstract

An integral relation for elements of the continued fraction of $\sqrt[3]{M}$ is presented.

Contents

Introduction

In the paper we prove a recurrence relation for elements and present a reliable hypothesis for remainders of the continued fraction of the cubic irrationality $\sqrt[3]{M}$. From the hypothesis it follows that the sequence of elements is unbounded. In particular, this gives the answer to the Kchinchin question [1] about the continued fractions of algebraic numbers.

For example, consider the continued fraction

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cdots}}}}}}}$$

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Let $r_0, r_1, r_2, \ldots$ be the sequence of remainders and let $a_0 = 1, a_1, a_2, \ldots$ be the sequence of elements of the continued fraction. Recall that

$$a_{k+1} = \left[\frac{1}{r_k}\right].$$

Direct calculations show that the sequence

$$\xi_k = \log_2(1 + r_k)$$

has the uniform distribution [2]. In spite of the chaotic behavior of the remainders all elements

3 1 5 1 1 4 1 1 8 1 14 1 10 2 1 4 12 2 3 2 1 3 4 1 1 2 14 3 12 1 ...

may be calculated by the recurrence relation

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - 2Q_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - 2Q_{k-1}^3)Q_{k-1}}\right],$$

where

$$\begin{array}{lll} P_{-1} = 1, & P_0 = 1, & P_k = a_k P_{k-1} + P_{k-2}, \\ Q_{-1} = 0, & Q_0 = 1, & Q_k = a_k Q_{k-1} + Q_{k-2}. \end{array}$$

0. An integral algorithm

Let $\omega = \sqrt[3]{M}$ be irrational. Consider the continued fraction

$$\omega \;=\; a_0 + r_0 \;=\; a_0 + \cfrac{1}{a_1 + r_1} \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + r_2}} \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + r_3}}} \;=\; \cdots$$

Here

$$\begin{aligned} a_k &= [1/r_{k-1}], \\ r_k &= 1/r_{k-1} - [1/r_{k-1}], \end{aligned}$$

where

$$\begin{aligned} a_0 &= [\omega], \\ r_0 &= \omega - [\omega]. \end{aligned}$$

We first give an integral algorithm for calculation $a_0, a_1, \ldots$.

Let us put

$$\begin{aligned} P_k &= a_k P_{k-1} + P_{k-2}, \\ Q_k &= a_k Q_{k-1} + Q_{k-2}, \end{aligned}$$

where

$$P_{-1} = 1 \quad Q_{-1} = 0 \qquad \text{and} \qquad P_0 = a_0 \quad Q_0 = 1$$

The identity

$$\omega = \frac{(a_k + r_k)P_{k-1} + P_{k-2}}{(a_k + r_k)Q_{k-1} + Q_{k-2}} \tag{1.0}$$

leads to the rational function

$$F_k(x) = \frac{P_{k-1}x + P_{k-2}}{Q_{k-1}x + Q_{k-2}}.$$

The function $F_k(x)$ is strictly monotone on $[0, \infty)$. Hence a_k is a unique integer $\alpha \geq 1$ such that

$$(M - F_k^3(\alpha)(M - F_k^3(\alpha + 1)) < 0. \tag{2.0}$$

We check the inequality (2.0) for $\alpha = 1, 2 \ldots$ and find a_k .

Of course, the algorithm is rather bulky. By identity (1.0), we have

$$a_k + r_k = -\frac{P_{k-2} - \omega Q_{k-2}}{P_{k-1} - \omega Q_{k-1}}.$$

Hence

$$a_k + r_k = -\frac{(P_{k-2}^3 - MQ_{k-2}^3)(P_{k-1}^2 + \omega P_{k-1}Q_{k-1} + \omega^2 Q_{k-1}^2)}{(P_{k-1}^3 - MQ_{k-1}^3)(P_{k-2}^2 + \omega P_{k-2}Q_{k-2} + \omega^2 Q_{k-2}^2)}.$$

For even k we have

$$\frac{P_{k-2}}{Q_{k-2}} < \omega < \frac{P_{k-1}}{Q_{k-1}}$$

and for odd k we have

$$\frac{P_{k-1}}{Q_{k-1}} < \omega < \frac{P_{k-2}}{Q_{k-2}}.$$

Hence for even k we obtain

$$-\frac{(P_{k-2}^3 - MQ_{k-2}^3)Q_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-2}^2} < a_k + r_k < -\frac{(P_{k-2}^3 - MQ_{k-2}^3)P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)P_{k-2}^2},$$

and for odd k we obtain

$$-\frac{(P_{k-2}^3 - MQ_{k-2}^3)P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)P_{k-2}^2} < a_k + r_k < -\frac{(P_{k-2}^3 - MQ_{k-2}^3)Q_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-2}^2}.$$

Let

$$b_k = \left[-\frac{(P_{k-2}^3 - MQ_{k-2}^3)Q_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-2}^2}\right] \quad \text{and} \quad c_k = \left[-\frac{(P_{k-2}^3 - MQ_{k-2}^3)P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)P_{k-2}^2}\right].$$

We see that $b_k \leq a_k \leq c_k$ for even k and $c_k \leq a_k \leq k_k$ for odd k. Therefore we may check the inequality (2.0) only for α between b_k and c_k. In the cases $k = 0$ and $k = 1$ we put $b_0 = 0, c_0 = \infty$ and $b_1 = \infty, c_1 = 0$.

Example. We apply the algorithm for $\sqrt[3]{4}$ and obtain

k	b_k	a_k	c_k
0	0	1	∞
1	∞	1	0
2	0	1	3
3	3	2	1
4	2	2	2
5	1	1	1
6	3	3	3
7	2	2	2
8	3	3	3

From the table we see that $a_k = b_k = c_k$ for $k \geq 4$.

Hypothesis. *Given M, there exists $n(M)$ such that $a_k = b_k = c_k$ or all $k \geq n(M)$.*

If the hypothesis is true, then

$$a_k = \left[-\frac{(P_{k-2}^3 - MQ_{k-2}^3)Q_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-2}^2} \right] = \left[-\frac{(P_{k-2}^3 - MQ_{k-2}^3)P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)P_{k-2}^2} \right]$$

for all sufficiently large k.

In the paper we prove the analogous formula

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}} \right].$$

Unfortunately, the last formula is not symmetric.

Remark. For $\sqrt[N]{M}$ we have

$$a_k = \left[-\frac{(P_{k-2}^N - MQ_{k-2}^N)Q_{k-1}^{N-1}}{(P_{k-1}^N - MQ_{k-1}^N)Q_{k-2}^{N-1}} \right] = \left[-\frac{(P_{k-2}^N - MQ_{k-2}^N)P_{k-1}^{N-1}}{(P_{k-1}^N - MQ_{k-1}^N)P_{k-2}^{N-1}} \right].$$

1. A cubic Diophantine equation

Theorem 1 *Let $A^3L \neq B^3N$. The Diophantine equation*

$$NX^3 - LY^3 = AX - BY$$

has at most three relatively prime positive solutions.

Proof. Let relatively prime $X \geq 1$ and $Y \geq 1$ satisfy the equation. From

$$(NX^2 - A)X = (LY^2 - B)Y$$

it follows that

$$\begin{aligned} NX^2 - A &= \lambda Y, \\ LY^2 - B &= \lambda X \end{aligned}$$

for an integer λ. Hence

$$BNX^2 - ALY^2 = \lambda BY - \lambda AX.$$

From

$$(BNX + \lambda A)X = (ALY + \lambda B)Y$$

it follows that

$$\begin{aligned} BNX + \lambda A &= \mu Y, \\ ALY + \lambda B &= \mu X \end{aligned}$$

for an integer μ. Hence

$$B^2NX - A^2LY = \mu BY - \mu AX.$$

From

$$(B^2N + \mu A)X = (A^2L + \mu B)Y$$

it follows that

$$\begin{aligned} X &= \gamma(A^2L + \mu B), \\ Y &= \gamma(B^2N + \mu A) \end{aligned}$$

for an integer γ. Since X are Y relatively prime, we have $\gamma = \pm 1$. Since X and Y are positive, we have either $\gamma = -1$ or $\gamma = 1$. From the input Diophantine equation we obtain

$$\mu^3 + 1 = 3ABNL\mu + A^3NL^2 + B^3N^2L.$$

Hance the Diophantine equation has at most three relatively prime positive solutions. •

Remark. Let $N \geq 1$ and $L \geq 1$. By the inequality

$$|\mu| \leq |A|^3NL^2 + |B|^3N^2L + 1,$$

we easily estimate X and Y.

2. Squares of numerators

Theorem 2 *For any $N \geq 1$ and A the congruence*

$$NP_k^2 \equiv A(\mathrm{mod} Q_k)$$

holds for a finite number of $k = 1, 2, \ldots$.

Proof. Assume that $k_1 < k_2 < \cdots$ and that

$$NP_k^2 = V_kQ_k + A \tag{1.2}$$

for all $k = k_1, k_2, \ldots$. It is known that

$$\omega = \frac{P_k}{Q_k} + \frac{\theta_k}{Q_k Q_{k+1}},$$

where $-1 < \theta_k < 0$ for odd k and $0 < \theta_k < 1$ for even k. From

$$N\frac{P_k^2}{Q_k^2} = \frac{V_k}{Q_k} + \frac{A}{Q_k^2}$$

and

$$\omega^2 = \frac{P_k^2}{Q_k^2} + 2\frac{P_k\theta_k}{Q_k^2 Q_{k+1}} + \frac{\theta_k^2}{Q_k^2 Q_{k+1}^2}$$

it follows that

$$N\omega^2 Q_k = V_k + \frac{A}{Q_k} + 2N\frac{P_k\theta_k}{Q_k Q_{k+1}} + N\frac{\theta_k^2}{Q_k Q_{k+1}^2}. \tag{2.2}$$

Multiplying (2.2) by

$$\omega Q_k = P_k + \frac{\theta_k}{Q_{k+1}},$$

we obtain

$$NMQ_k^2 = V_k P_k + B_k, \tag{3.2}$$

with an integer

$$B_k = A\frac{P_k}{Q_k} + \frac{3NP_k^2}{Q_k Q_{k+1}}\theta_k + \frac{3NP_k}{Q_k Q_{k+1}^2}\theta_k^2 + \frac{N}{Q_k Q_{k+1}^3}\theta_k^3.$$

Since

$$\frac{P_k}{Q_k} \le \omega + \frac{1}{Q_k Q_{k+1}} \le \omega + 1,$$

we have

$$|B_k| \le |A|(\omega + 1) + N(3\omega^2 + 9\omega + 7)$$

for all k. Multiplying (1.2) by P_k, we obtain

$$NP_k^3 = V_k P_k Q_k + AP_k. \tag{4.2}$$

Multiplying (3.2) by Q_k, we obtain

$$NMQ_k^3 = V_k P_k Q_k + B_k Q_k. \tag{5.2}$$

Subtracting (5.2) from (4.2), we obtain

$$NP_k^3 - NMQ_k^3 = AP_k - B_k Q_k.$$

The sequence $B_{k_1}, B_{k_2}, \ldots$ is bounded. Hence there exists an integer B such that

$$NP_k^3 - NMQ_k^3 = AP_k - BQ_k$$

for infinitely many k. Since $A^3 \ne B^3 M$, and since $P_k \ge 1$ and $Q_k \ge 1$ are relatively prime, we have a contradiction with Theorem 1. The theorem is proved. •

3. A recurrence relation

Theorem 3 *Given $\omega = \sqrt[3]{M}$, there exists $n(M)$ such that*

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}} \right]$$

for all $k \geq n(M)$.

Proof. From

$$\omega = \frac{(a_k + r_k)P_{k-1} + P_{k-2}}{(a_k + r_k)Q_{k-1} + Q_{k-2}}$$

it follows that

$$a_k + r_k = -\frac{P_{k-2} - \omega Q_{k-2}}{P_{k-1} - \omega Q_{k-1}}.$$

Hence

$$a_k + r_k = -\frac{(P_{k-2} - \omega Q_{k-2})(P_{k-1}^2 + \omega P_{k-1}Q_{k-1} + \omega^2 Q_{k-1}^2)}{P_{k-1}^3 - MQ_{k-1}^3}.$$

From

$$Q_{k-1}P_{k-2} - Q_{k-2}P_{k-1} = (-1)^{k-1}$$

we obtain

$$a_k + r_k = -\frac{P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2 + (-1)^{k-1}(\omega P_{k-1} + \omega^2 Q_{k-1})}{P_{k-1}^3 - MQ_{k-1}^3}.$$

From

$$\omega = \frac{P_{k-1}}{Q_{k-1}} + \frac{\theta_{k-1}}{Q_{k-1}Q_k},$$

where $-1 < \theta_k < 0$ for odd k and $0 < \theta_k < 1$ for even k, we obtain

$$\omega P_{k-1} + \omega^2 Q_{k-1} = \frac{2P_{k-1}^2}{Q_{k-1}} + \frac{3P_{k-1}\theta_{k-1}}{Q_{k-1}Q_k} + \frac{\theta_{k-1}^2}{Q_{k-1}Q_k^2}.$$

Hence

$$a_k + r_k = -\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}(2P_{k-1}^2 + \Delta_k)}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}}, \tag{1.3}$$

where

$$\Delta_k = \frac{3P_{k-1}\theta_{k-1}}{Q_k} + \frac{\theta_{k-1}^2}{Q_k^2}.$$

Let us put

$$C = 3\omega + 4.$$

Then

$$|\Delta_k| \leq C$$

for all k. Let us write

$$2P_k^2 = V_k Q_k + A_k,$$

where $0 \le A_k \le Q_k - 1$. By Theorem 2, there exists $n(M)$ such that

$$C + 1 \le A_{k-1} \le Q_{k-1} - (C+1)$$

for all $k \ge n(N)$. Let us rewrite (1.3) in the form

$$a_k + r_k = \frac{(-1)^{k-1}(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + 2P_{k-1}^2 + \Delta_k}{(-1)^k(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}}.$$

We note that the denominator is positive for all k. Consider

$$a_k + r_k = \frac{(-1)^{k-1}(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2) + V_{k-1} + (A_{k-1} + \Delta_k)/Q_{k-1}}{(-1)^k(P_{k-1}^3 - MQ_{k-1}^3)}.$$

It is clear that

$$1 \le A_{k-1} + \Delta_k \le Q_{k-1} - 1$$

for all $k \ge n(M)$. If P and $Q \ge 1$ are integers, then

$$\left[\frac{P+\delta}{Q}\right] = \left[\frac{P}{Q}\right]$$

for all $0 \le \delta < 1$. Hence for all $k \ge n(M)$ the integer

$$a_k = \left[\frac{(-1)^{k-1}(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2) + V_{k-1} + (A_{k-1} + \Delta_k)/Q_{k-1}}{(-1)^k(P_{k-1}^3 - MQ_{k-1}^3)}\right]$$

is equal to the integer

$$\left[\frac{(-1)^{k-1}(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2) + V_{k-1} + A_{k-1}/Q_{k-1}}{(-1)^k(P_{k-1}^3 - MQ_{k-1}^3)}\right].$$

The theorem is proved. •

Hypothesis. *Given* $\sqrt[N]{M}$, *there exists* $n(N, M)$ *such that*

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^{N-1} - MQ_{k-2}Q_{k-1}^{N-1})Q_{k-1} + (-1)^{k-1}(N-1)P_{k-1}^{N-1}}{(P_{k-1}^N - MQ_{k-1}^N)Q_{k-1}}\right]$$

for all $k \ge n(N, M)$.

4. Numbers $M = 2, 9, 11, 12, 13, 14, 16, 19, 20, \ldots$

Let $n(M)$ be a minimal natural such that

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}} \right]$$

for all $k \geq n(M)$.

Theorem 4 *Let n satisfy the inequality $P_{n-1} \geq 10^8 M^4$. Assume that*

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}} \right]$$

for all $1 \leq k \leq n-1$. Then $n(M) = 1$.

Доказательство. Let us write

$$2P_k^2 = V_k Q_k + A_k,$$

where $0 \leq A_k \leq Q_k - 1$. By the proof of Theorem 3, the identity

$$a_k = \left[-\frac{(P_{k-2}P_{k-1}^2 - MQ_{k-2}Q_{k-1}^2)Q_{k-1} + (-1)^{k-1}2P_{k-1}^2}{(P_{k-1}^3 - MQ_{k-1}^3)Q_{k-1}} \right]$$

follows from the inequality

$$3\omega + 5 \leq A_{k-1} \leq Q_{k-1} - (3\omega + 5),$$

where $k \geq n$.

Assume that

$$0 \leq A_{k-1} \leq 3\omega + 5 \quad \text{or} \quad Q_{k-1} - (3\omega + 5) \leq A_{k-1} \leq Q_{k-1}$$

for some $k \geq n$. Then

$$2P_{k-1}^2 = U_{k-1}Q_{k-1} + A,$$

where

$$|A| \leq 3\omega + 5.$$

By the proof of Theorem 2, there exists an integer B such that

$$|B| \leq (3\omega + 5)(\omega + 1) + 2(3\omega^2 + 9\omega + 7) = 9\omega^2 + 26\omega + 19$$

and

$$2P_{k-1}^3 - 2MQ_{k-1}^3 = AP_{k-1} - BQ_{k-1}.$$

Hence P_{k-1} and Q_{k-1} satisfy the Diophantine equation

$$2X^3 - 2MY^3 = AX - BY.$$

By the proof of Theorem 2, we have

$$P_{k-1} \leq 10^8 M^4 - 1.$$

Since $k \geq n$, we have a contradiction. The theorem is proved. $\bullet$

By direct calculation, we see that $n(M) = 1$ for

$M = 2, 9, 11, 12, 13, 14, 16, 19, 20, 28, 29, 30, 32, 33, 35, 36, 38, 39, 40, 43, 44, 45, 46, 47, 50, \ldots$

5. A hypothesis about remainders

Let $x_1, x_2, \ldots$ be a sequence on $[0,1]$. Assume that

$$F(t) = \lim_{n\to\infty} \frac{m_n(t)}{n}$$

for all $0 \le t \le 1$, where $m_n(t)$ is a number of $x_1, \ldots, x_n$ such that $0 \le x_k \le t$. The function

$$f(t) = \frac{d}{dt} F(t)$$

is called the density of the sequence $x_1, x_2, \ldots$. If $f(t) = 1$, then $x_1, x_2, \ldots$ has the uniform distribution. For example, let x be irrational, that the sequence

$$x_n = nx - [nx]$$

has the uniform distribution.

Consider the remainders $r_0, r_1, r_2, \ldots$ of the continued fraction

$$\sqrt[3]{M} \;=\; a_0 + r_0 \;=\; a_0 + \cfrac{1}{a_1 + r_1} \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + r_2}} \;=\; a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + r_3}}} \;=\; \cdots$$

Hypothesis. *The sequence*

$$\xi_n = \log_2(1 + r_n)$$

has the uniform distribution.

Example. If we calculate 1001 elements $\xi_0, \xi_1, \ldots, \xi_{1000}$ for $\sqrt[3]{2}$, then

$$\frac{1}{1001} \sum_{k=0}^{1000} \xi_k = 0.5015... \approx \frac{1}{2},$$

$$\frac{1}{1001} \sum_{k=0}^{1000} \xi_k^2 = 0.3321... \approx \frac{1}{3},$$

$$\frac{1}{1001} \sum_{k=0}^{1000} \xi_k^3 = 0.2470... \approx \frac{1}{4},$$

$$\frac{1}{1001} \sum_{k=0}^{1000} \xi_k^4 = 0.1959... \approx \frac{1}{5}.$$

Analogous results we have for other $\sqrt[3]{M}$.

Remark. The hypothesis is naturally connected with the known Gauss theorem [1]. Let

$$x = r_0 = \frac{1}{a_1 + r_1} = \frac{1}{a_1 + \frac{1}{a_2 + r_2}} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + r_3}}} = \cdots$$

where $0 \le x \le 1$. Let x be a random variable with the uniform distribution on $[0,1]$. Then r_n is a random variable on $[0,1]$. Let $f_n(t)$ be the probability density of r_n. By the Gauss theorem

$$\lim_{n\to\infty} f_n(t) = \frac{1}{\ln 2}\frac{1}{1+t}.$$

This means that the random variables

$$\xi_n = \log_2(1 + r_n)$$

converge to the uniform distribution.

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