Median Logic

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Abstract
Median logic introduced here is a minimal intermediate logic that combines classical properties in its propositional part and intuitionistic properties for derivations not containing propositional symbols. A sequent calculus and other formulations are presented for median logic. Cut elimination is proven for the sequent calculus formulation. Constrained Kripke structures are introduced for modeling median logic. The extent of the disjunction and existence properties is investigated.

Keywords: intermediate logic, classical logic, intuitionistic logic, sequent calculus, cut elimination, Kripke structures, disjunction property, existence property

Introduction

The difference between intuitionistic and classical logic is in the Aristotelian law of excluded middle (LEM) and its consequences: $A \vee \neg A$. From the intuitionistic point of view, LEM is not acceptable because there is no general method for deciding whether A(x) is true or false for arbitrary *x* if x ranges over an infinite domain. From the classical point of view, LEM is a valid and very important principle used everywhere in mathematics. As Hilbert stated [Hi]: “Taking the principle of excluded middle from the mathematician would be the same, proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether.”

Which of the two logics - classical and intuitionistic - is better? This rhetorical question has been around for about a century. The matter of fact is that each has certain advantages over the other in certain dimensions. Both intuitionistic and classical logic have issues, too. Classical logic has decent models defined with using truth tables, but existence proofs are not constructive in it. Intuitionistic logic has the disjunction and existence properties and thus constructive proofs but the propositional fragment of intuitionistic logic while being finite does not have precise finite models.

Is it possible to combine the best of both worlds? Research in this domain was initiated by Godel and Tarski almost as soon as intuitionistic logic emerged. First of all, classical logic can be interpreted within intuitionistic one via Godel-Gentzen-Kolmogorov’s double negation translation. Since derivable intuitionistic formulas constitute a subset of derivable classical formulas, the focus of this research has been on investigating properties of the logics lying between the two. Such logics are called intermediate or superintuitionistic. Intermediate logics are usually defined by adding one or more axiom schemas weaker than LEM to intuitionistic logic. For example, the Dummett logic is defined by adding the axiom schema: $(A \supset B) \vee (B \supset A)$. See [Fi] for an overview of intermediate logics. Many intermediate logics have the disjunction property [FM, Ma].

Recently, research in the area of combining features of classical and intuitionistic logic has shifted towards logics containing two different variants of connectives. One of them is intuitionistic, and the other is classical [Kr] Fibring logics is the most noticeable technique in this research [Ga]. This approach is not quite intuitive though. It is generally not clear where to apply which of the two paired connectives. Besides, this approach in its original form suffers the collapsing problem - merely joining intuitionistic and classical axiomatizations with addition of some interaction axioms makes the combination logic collapse into classical logic [FH, SRC].

Let us have a fresh look at this problem of combining features of intuitionistic and classical logic into one without additional connectives. In contrast to traditional intermediate logic research, we deliberately segregate propositional logic because the classical assumption that every prime propositional formula is either true or false but not both is so natural. Extending this assumption onto prime predicate expressions is not quite natural though as discovered by intuitionists. Basically, we stand upon differences between formulations for finite and infinite parts of logic.

Note that Brouwer [He] criticized LEM only because it was abstracted from finite situations and extended without justification to statements about infinite collections. As Weyl [We] states in reference to Brouwer: “classical logic was abstracted from the mathematics of finite sets and their subsets”. Propositional logic can be considered a part of the mathematics of finite sets because of availability of finite models using truth tables. Given the decidability of prime propositional formulas, LEM is a desirable property for propositional logic. The disjunction property is not essential for propositional logic due to its finiteness. The disjunction and existence properties are perhaps the two most desirable properties for quantifier formulas that can map to infinite domains. The disjunction and existence properties are inherently intuitionistic. They lead to constructiveness of proofs in infinite domains. Now it is clear that we have described logic whose propositional fragment is classical whereas the intrinsically predicate part is intuitionistic.

Let us find out to what extent this particular goal of combining classical propositional logic with intuitionistic predicate logic can be achieved. This goal is only achievable with partially sacrificing the substitution property because we want $A\vee\neg A$ be derivable and $\exists x\, P(x)\vee\neg \exists x\, P(x)$ not derivable. Note though that substitution of predicate formulas for a propositional symbol is not the same as substitution of a propositional formula since the former substitution means a leap forward to infinite domains. Therefore, limiting the scope of substitutions to propositional formulas is not unreasonable if we deliberately segregate the finite propositional part from the rest.

Note that the simplest solution to combining intuitionistic and classical logic in this way is just to combine all theorems of classical propositional calculus and those of intuitionistic first-order calculus. Then we would have all the classical properties in the propositional part and all intuitionistic properties in the predicate part. It is an ugly solution though. Let us look at a classically provable propositional formula that does not hold intuitionistically. It could be, for instance, $A\vee\neg A$ where A is a propositional symbol. This formula would hold in the combined logic as well. Although, $(A\vee\neg A) \vee \exists x\, P(x)$ would not hold in the combined logic because it does not hold in intuitionistic logic. This case illustrates that the combined logic would not even have the following property: $F\supset F\vee G$. The propositional and predicate fragments of this logic are totally disjoint, and there is a sharp edge between the two. It would not have a common set of inference rules applicable to both propositional and predicate fragments.

Thus, we confine ourselves to logics closed under modus ponens and closed under propositional substitution. Our goal is to define a logic that is as much intuitionistic as possible but still includes classical propositional logic. Basically, we want to define a minimal extension of intuitionistic logic. We will define such logic by one set of inference rules. These rules are applicable to both propositional and predicate formulas but one axiom schema (LEM) will be applicable to propositional formulas only. Since this logic is minimal, the niceties of intuitionistic logic are preserved to the maximum extent possible. This definition is based on sequent calculi. Later, equivalent formulations for natural deduction and Hilbert-style calculi will be given.

We define a minimal intermediate logic that coincides with classical logic in its propositional part and coincides with intuitionistic logic on the set of formulas either not containing propositional symbols or having occurrences of the same polarity for every propositional symbol. Therefore, this logic possesses the disjunction and existence properties for the formulas whose every propositional symbol has occurrences of the same polarity. Additionally in this logic, the disjunction property holds for disjunctions whose disjuncts do not share any propositional symbols. Some additional classes of formulas for which the existence property hold are also identified.

This logic will be called median logic because, in a sense, it lies right in the middle between intuitionistic and classical logic. The minimality of median logic is critical because it implies that no other extension covering classical propositional logic can be made more 'intuitionistic' than median logic. Whereas supersets of median logic are less intuitionistic, its subsets are not fully classical in the propositional part. Considering the formulas that are derivable classically but not intuitionistically, median logic comprises all of them among purely propositional formulas, none of them among purely predicate formulas, and somewhat 'half' of them in between.

Median logic has a balanced mix of classical and intuitionistic characteristics. Derivable propositional and first-order formulas smoothly blend in median logic. The cut elimination theorem holds for median logic. Median logic is sound and complete with respect to constrained Kripke structures introduced here. In contrast to fibring logics, median logic does not have paired connectives, which makes it much more natural and intuitive. It does not suffer the collapsing problem either.

Note that research with similar motivations and somewhat similar goals was recently done by H. Kurokawa [Ku]. He investigated propositional logic with two types of prime formulas: classical and intuitionistic. And LEM is applicable to classical formulas only. From the perspective of Kurokawa's research, median logic is a first-order intuitionistic logic whose prime propositional formulas are classical.

Definition of Median Logic

We will call 0-place predicate symbols propositional symbols in order to distinguish them from the rest. As we see later, formulas containing only propositional symbols will have very different properties than the rest of formulas because the former constitute a finite fragment of the logic under investigation. We use the phrase 'predicate symbol' only in application to symbols whose arity is more than zero.

We use letters P, Q, R, .. for denoting propositional and predicate symbols. Letters A, B, C, ,,, will denote formulas. Capital Greek letters will denote (possibly empty) sequences of formulas.

Definition. A formula is called propositional if it contains only propositional symbols. A formula is called purely predicate if all its symbols are predicate symbols. A formula is called mixed if it contains both propositional and predicate symbols.

First, we present two equivalent Gentzen-style formulations of median logic. Both Gentzen-style variants of median logic will be called LJ+. A Hilbert-style formulation for median logic and a formulation in the style of natural deduction will be presented later.

We use the definition of LJ from [Ge]. A sequent is an expression $\Gamma \vdash \Lambda$ where Γ is a (possibly empty) sequence of formulas and Λ is either a formula or an empty sequence. Γ is called antecedent, Λ is called consequent. The only axiom schema of LJ is:
$A \vdash A$

The rules of inference are: thinning, contraction, exchange, cut, pairs of rules for negation, implication, existential quantifier, and universal quantifier, and triples of rules for conjunction and disjunction. One of the rules of each pair operates on antecedents whereas the other(s) operates on consequents. These rules are repeated below.

Thinning:
$$\frac{\Gamma \vdash \Lambda}{A, \Gamma \vdash \Lambda} \qquad \frac{\Gamma \vdash}{\Gamma \vdash A}$$

Contraction:
$$\frac{A, A, \Gamma \vdash \Lambda}{A, \Gamma \vdash \Lambda}$$

Exchange:
$$\frac{\Pi, A, B, \Gamma \vdash \Lambda}{\Pi, B, A, \Gamma \vdash \Lambda}$$

Cut:
$$\frac{\Gamma \vdash C \qquad C, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Lambda}$$

&:
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\&B} \qquad \frac{A, \Gamma \vdash \Lambda}{A\&B, \Gamma \vdash \Lambda} \qquad \frac{B, \Gamma \vdash \Lambda}{A\&B, \Gamma \vdash \Lambda}$$

$\vee$:
$$\frac{A, \Gamma \vdash \Lambda \qquad B, \Gamma \vdash \Lambda}{A \vee B, \Gamma \vdash \Lambda} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$\neg$:
$$\frac{A, \Gamma \vdash}{\Gamma \vdash \neg A} \qquad \frac{\Gamma \vdash A}{\neg A, \Gamma \vdash}$$

$\supset$:
$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \quad B, \Pi \vdash \Lambda}{A \supset B, \Gamma, \Pi \vdash \Lambda}$$

$\forall$:
$$\frac{\Gamma \vdash F(a)}{\Gamma \vdash \forall x\, F(x)} \qquad \frac{F(a), \Gamma \vdash \Lambda}{\forall x\, F(x), \Gamma \vdash \Lambda}$$

$\exists$:
$$\frac{\Gamma \vdash F(a)}{\Gamma \vdash \exists x\, F(x)} \qquad \frac{F(a), \Gamma \vdash \Lambda}{\exists x\, F(x), \Gamma \vdash \Lambda}$$

Here variable a occurs free in F, F(x) is obtained from F(a) by replacing all free occurrences of a by x. This variable (a) in the $\forall$-succedent rule and in the $\exists$-antecedent rule is called eigenvariable. It must not occur in the lower sequents of the respective rules.

Typically, two constants - $\top$ and $\perp$ - are introduced as well. $\top$ denotes a ‘valid’ formula whereas $\perp$ denotes an ‘unsatisfiable’ formula. Formally, these constants are introduced by the following axioms:

$$\vdash \top \qquad\qquad \perp \vdash$$

Definition 1 of LJ+. LJ+ is LJ extended with one additional axiom schema:

$$\vdash N \vee \neg N$$

Only propositional formulas are allowed in this axiom schema. Like in LJ, every sequent in LJ+ has not more than one formula in succedent.

Definition 2 of LJ+. LJ+ is LJ extended with one additional rule:

$$\frac{N, \Gamma \vdash \Lambda \qquad \neg N, \Gamma \vdash \Lambda}{\Gamma \vdash \Lambda}$$

It will be called neutralization rule. Only propositional formulas N are allowed in the neutralization rule. Every sequent in LJ+ has not more than one formula in succedent.

Theorem. The two definitions of LJ+ are equivalent, that is, every sequent derivable by one of the definitions is also derivable by the other.

Proof. To transform a derivation by definition 2 into one by definition 1, it is only necessary to transform all applications of the neutralization rule into the following:

$$\dfrac{\vdash N \vee \neg N \qquad \dfrac{N, \Gamma \vdash \Lambda \qquad \neg N, \Gamma \vdash \Lambda}{N \vee \neg N, \Gamma \vdash \Lambda}}{\Gamma \vdash \Lambda}\ \text{cut}$$

To transform a derivation by definition 1 into one by definition 2, let us replace all axioms $\vdash N \vee \neg N$ with the following derivation using the neutralization rule:

$$\dfrac{\dfrac{N \vdash N}{N \vdash N \vee \neg N} \qquad \dfrac{\neg N \vdash \neg N}{\neg N \vdash N \vee \neg N}}{\vdash N \vee \neg N}\ \text{neutralization}$$

♦

Note that every LJ derivation is also a valid LJ+ derivation. Therefore, every formula derivable in LJ is also derivable in LJ+. Also, every formula derivable in LJ+ is also derivable in LK. LK is Gentzen-style formulation of classical logic. Gentzen [Ge] alternatively defines LK as LJ (i.e. all sequent have no more than one formula in the succedent) plus axiom schema |-- $A \lor \neg A$ that is applicable to any formula A. LJ+ is consistent because the set of derivable formulas of LJ+ is a subset of those of LK, and LK is known to be consistent.

Speaking of intermediate logics, it is usually required that they are closed under modus ponens and under propositional substitution. LJ+ apparently meets these requirements. A well-know proof of sequent calculus closeness under modus ponens is done by applying cut to both formulas in the antecedent of $A \supset B$, A |-- B. This sequent is derived in one step by application of the ⊃-antecedent rule. This proof apparently works for LJ+ too. Let us look at arbitrary derivation in LJ+ and a propositional symbol occurring in the derivation. If we replace all occurrences of this propositional symbol by any propositional formula, this derivation will remain a correct LJ+ derivation, thus closeness under propositional substitution. In contrast to classical and intuitionistic logic, median logic is not closed under substitution of predicate formulas for propositional symbols.

Median logic is not closed under substitution of predicate formulas for propositional symbols though. In my opinion, it is quite acceptable because when we replace a propositional symbol by a quantifier formula, we cross a significant divide. It is the divide between finite and infinite. Thus by substituting a predicate formula for a propositional symbol we change the nature of the formula. Recall also that critique of LEM by intuitionists relates to application of LEM to infinite domains but not finite ones. Recognition of the difference between a finite propositional realm and its infinite predicate-containing complement is a major premise of this work.

There exist formulas provable in LJ+ but not provable in LJ. $R \lor \neg R$ for propositional symbol R is an example of such formula holding LJ+. See [Ge] for the proof that $R \lor \neg R$ does not hold in LJ. Also, there exist formulas provable in LK but not in LJ+. This will be proven later - see section Intuitionistic Nature of Median Logic. $\exists x\ P(x) \lor \neg\ \exists x\ P(x)$ is an example of a classically provable formula that does not hold in LJ+.

Sample Theorems

This section illustrates what kind of derivations can be conducted in LJ+. Three simple examples are presented below. More sample derivations can be found in sections ‘Classical Nature of Median Logic’ and ‘Disjunction and Existence Properties’.

1. Definition 2 Example
Note that as a part of the following proof, we derive double negation elimination for propositional formulas.

```
                  ¬A |-- ¬A
                  ------------
A |-- A           ¬A, ¬¬A |--
--------          ----------------
A, ¬¬A |-- A      ¬A, ¬¬A |-- A
--------------------------------------- neutralization
```

$$
\dfrac{\dfrac{\neg\neg A \vdash A}{\neg\neg A \vdash A \vee \forall x\, B(x)} \qquad \dfrac{\forall x\, B(x) \vdash \forall x\, B(x)}{\forall x\, B(x) \vdash A \vee \forall x\, B(x)}}{\dfrac{\neg\neg A \vee \forall x\, B(x) \vdash A \vee \forall x\, B(x)}{\vdash (\neg\neg A \vee \forall x\, B(x)) \supset (A \vee \forall x\, B(x))}}
$$

2. Definition 1 Example
In this example, a simple LJ derivation is omitted.

$$
\dfrac{\dfrac{\vdash A \vee \neg A}{\forall x\, B(x) \vdash A \vee \neg A} \qquad \dfrac{\dots}{\forall x\, B(x) \vdash \exists x\, B(x)}}{\dfrac{\forall x\, B(x) \vdash (A \vee \neg A)\ \&\ \exists x\, B(x)}{\vdash \forall x\, B(x) \supset ((A \vee \neg A)\ \&\ \exists x\, B(x))}}
$$

3. Definition 2 Example

$$
\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A \vee \forall x\, P(x)}}{A \vdash (A \vee \forall x\, P(x)) \vee (\neg A \vee \forall x\, Q(x))} \qquad \dfrac{\dfrac{\neg A \vdash \neg A}{\neg A \vdash \neg A \vee \forall x\, Q(x)}}{\neg A \vdash (A \vee \forall x\, P(x)) \vee (\neg A \vee \forall x\, Q(x))}}{\vdash (A \vee \forall x\, P(x)) \vee (\neg A \vee \forall x\, Q(x))}\ \text{neutralization}
$$

Classical Nature of Median Logic

Theorem. The propositional fragment of LJ+ is equivalent to classical propositional logic, that is, to the propositional fragment of LK. There exist mixed formulas provable in LJ+ but not provable in LJ.

Proof. LK can be alternatively defined as LJ (with all sequent having no more than one formula in the succedent) plus axiom schema $\vdash A \vee \neg A$ that is applicable to any formula A. Given that, note that every LK derivation (by the above definition) containing only propositional formulas is also a valid LJ+ derivation. As far as the second statement of this theorem is concerned, consider formula $(R \vee \neg R) \vee P(x)$ in which both R and P are atomic. It is apparently derivable in LJ+ because $A \vee \neg A$ is derivable in LJ+. Assume it is derivable in LJ. Due to the disjunction property, either $R \vee \neg R$ or $P(x)$ should be derivable in LJ. $P(x)$ could not be derivable in LJ because it could not be the bottom sequent of any rule except thinning of $\vdash$ which in turn is not derivable in LJ. It is known that $R \vee \neg R$ is not derivable in LJ either. ♦

The following theorem is characteristic of classical logic, and it does not hold in intuitionistic logic. Interestingly, this theorem holds in LJ+. If R is a propositional symbol occurring in formula A, then let $A\{R|F\}$ denote the formula obtained from A by replacing all occurrences of R with formula F.

Theorem. If R is a propositional symbol occurring in formula A, then A is derivable in LJ+ if and only if both $A\{R|\top\}$ and $A\{R|\bot\}$ are derivable in LJ+.

Proof. The 'if' part is proved by replacing all occurrences of R by $\top$ and $\bot$ in A's derivation. Apparently, the derivation remains intact. Now consider derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ and the occurrences of $\top$ and $\bot$ that replaced R. The constants $\top$ and $\bot$ introduced in derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ by other means than the axioms

$\vdash \top$ and $\bot \vdash$

can be replaced by R retaining the derivation intact because the only other means is thinning. If $\top$ is introduced by axiom $\vdash \top$ in derivation of $A\{R|\top\}$, let us replace this axiom by axiom

$R \vdash R$

If $\bot$ is introduced by axiom $\bot \vdash$ in derivation of $A\{R|\bot\}$, let us replace this axiom by

$R, \neg R \vdash$

which is derivable from the above axiom in one step.

After that, let us transform derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ so that R ($\neg$R) is added as the rightmost formula to the antecedents of all sequents below the replaced axioms for $\top(\bot)$ and to the antecedent of their counterpart sequents in the upper parts of rules. This transformation is done recursively top down to the endsequent. If a rule has one upper sequent and R ($\neg$R) has been already added as the rightmost formula to the antecedent of its upper sequent, then let us add R ($\neg$R) as the rightmost formula in the antecedent of the lower sequent. Apparently, this rule remains intact. If a rule has two upper sequents and R ($\neg$R) has been already added as the rightmost formula to the antecedent of one of its upper sequents, then let us add R ($\neg$R) to the counterpart upper sequent by injection of thinning and exchanges in order to make R ($\neg$R) the rightmost formula of the antecedent of this sequent. If we also add R ($\neg$R) as the rightmost formula to the antecedent of the lower sequent, then this rule remains intact. In case R ($\neg$R) has already been added to both upper sequents, let us add R ($\neg$R) as the rightmost formula to the antecedent of the lower sequent, and this rule remains intact as well. As a result of this transformation, both derivations under consideration remain correct. The endsequent in the derivation of $A\{R|\top\}$ will have one formula R in the antecedent whereas the endsequent in the derivation of $A\{R|\bot\}$ will have one formula $\neg$R in the antecedent. These two derivations are merged by application of the neutralization rule to R and $\neg$ R. The resulting endsequent is A. ♦

One other interesting classical feature of LJ+ is that $N \supset F$ is equivalent to $\neg N \vee F$ for any formula F and any propositional formula N. This is proved directly by constructing simple derivations for the implications in both directions. In the following two figures presenting these derivations, double lines denote application of more than one rule.

```
N |-- N     F |-- F
-------------------
N⊃F, N |-- F                  ¬N |-- ¬N
========                      =======
N, N⊃F |-- ¬NVF               ¬N, N⊃F |-- ¬NVF
--------------------------------------------------------------
          N⊃F |-- ¬NVF
```

```
N |-- N                    F |-- F
=====                      ====
¬N, N |--  F               F, N |--  F
---------------------------------------
        ¬N∨F, N |-- F
        ----------------
        ¬N∨F |-- N⊃F
```

Median logic possesses features of intuitionistic logic as well. We will discuss intuitionistic nature of median logic later when we establish other facts necessary for the respective proofs.

Median Logic Properties

In general, theorems characteristic for both classical and intuitionistic logic hold in median logic as well. For example, the replacement theorem and the deduction theorem [Kl] are such. We are not going to present these proofs here because they would be similar to well-known proofs for classical and intuitionistic logics.

Since one of axioms applies to propositional formulas only, it is important that median logic does not have a sharp edge between propositional formulas and formulas containing predicate symbols. Median logic possesses a smoothness property with respect to the border between its classical propositional part and the rest. For example, if A is classically provable propositional formula (which may not hold intuitionistically), then $A \vee \exists x\, P(x)$ does hold in LJ+ (but may not hold in LJ). Here is a well-known definition of polarity.

Definition. Positive (negative) sub-formula occurrences in propositional formula A are defined recursively by the following rules:
- A itself is positive.
- If sub-formula $B \vee C$ or $B\&C$ is positive (negative), then B is positive (negative).
- If sub-formula $\neg B$ is positive (negative), then B is negative (positive).
- If sub-formula $B \supset C$ is positive (negative), then B is negative (positive) and C is positive (negative).
- If sub-formula $\forall B(x)$ is positive (negative), then $B(x)$ is positive (negative).

Suppose B is a sub-formula of A. Consider a particular occurrence of B in A. Let $A_{B|C}$ denote the formula obtained from A by replacing the given occurrence of B by C. The following is a well-known theorem holding both classically and intuitionistically. Its proof for median logic is similar to those for classical and intuitionistic logics. This theorem ensures smoothness with respect to sub-formula replacements.

Theorem. Let C denote a predicate formula. If B is a positive sub-formula occurrence in formula A, and both A and $B \supset C$ are derivable in LJ+, then $A_{B|C}$ holds in LJ+. If B is a negative sub-formula occurrence in formula A, and both A and $C \supset B$ hold in LJ+, then $A_{B|C}$ holds in LJ+.

Corollary. Let C denote a predicate formula. If B is a positive sub-formula occurrence in formula A, and A holds in LJ+, then $A_{B|B \vee C}$ holds in LJ+. If B is a negative sub-formula occurrence in formula A, and A holds in LJ+, then $A_{B|B\&C}$ holds in LJ+.

LJ+ does not have a sharp edge between purely predicate formulas and mixed formulas either.

The following theorem will be proven later in the section Intuitionistic Nature of Median Logic.

Definition. Occurrence of formula B in formula A is called strictly positive if there are no outer connectives for B except for & and $\vee$ and this occurrence of B is not in the scope of any quantifier in A.

Theorem. If A is classically provable purely predicate formula that is not derivable in LJ+, B is strictly positive sub-formula occurrence in A, and C is a propositional formula not derivable in LJ+, then $A_{B|B\vee C}$ does not hold in LJ+. If D is any propositional formula, then $A_{B|B\&D}$ does not hold in LJ+.

The following theorem confirms that median logic is a minimal extension of intuitionistic logic that includes classical propositional logic and is closed under modus ponens and propositional substitution. Therefore no other extension covering classical propositional logic can be made more 'intuitionistic' than median logic.

Lemma. If Formula F is derivable in LJ+, then $\Gamma \mid\!\!- F$ is derivable in LJ for some formula sequence Γ of the form $A_1\vee\neg A_1,\ldots,A_n\vee\neg A_n$ where $A_1,\ldots,A_n$ are propositional formulas.

Proof. Consider derivation of F in LJ+ by definition 1. Let $A_1\vee\neg A_1,\ldots,A_n\vee\neg A_n$ be all the right-hand sites of the LJ+ additional axioms present in the derivation. Let us replace these axioms with axioms $A_i\vee\neg A_i \mid\!\!- A_i\vee\neg A_i$, respectively. After that, let us inject thinnings and exchanges into this derivation. These thinnings and exchanges serve to augment antecedents of the original sequents with some of the formulas from the list $A_1\vee\neg A_1,\ldots,A_n\vee\neg A_n$ as the rightmost formulas in the antecedents. It is done as necessary to balance the pairs of antecedents in rules having two upper sequents. Once two upper sequents are balanced, the same formulas are added as the rightmost formulas to the antecedent of the lower sequent. This transformation is done recursively top down to the endsequent, and it helps retain a correct LJ+ derivation. This transformation is similar to one from the proof of the second theorem in the section Classical Nature of Median Logic. This modified LJ+ derivation does not have the additional axioms, and thus, it is a correct LJ derivation as well. The endsequent of the modified derivation is $A_1\vee\neg A_1,\ldots,A_n\vee\neg A_n \mid\!\!- F$. ♦

Theorem. If first-order logic L is an extension of LJ including classical propositional logic and L is closed under modus ponens and propositional substitution, then any formula F derivable in LJ+ is also derivable in L.

Proof. By the lemma, if F is derivable in LJ+, then $A_1\vee\neg A_1,\ldots,A_n\vee\neg A_n \mid\!\!- F$ is derivable in LJ for some propositional formulas $A_1,\ldots,A_n$. Note that $(A_1\vee\neg A_1)\&\ldots\&(A_n\vee\neg A_n)\supset F$ is also derivable in LJ. The proof of the latter is done by induction on n and by applying the &-antecedent, exchange and contraction rules. Formula $(A_1\vee\neg A_1)\&\ldots\&(A_n\vee\neg A_n)$ is derivable in L because L includes classical propositional logic. Since L is closed under modus ponens, F is also derivable in L. ♦

Cut Elimination

The cut rule can be eliminated from derivations in LJ+ by definition 2. Whereas cut can be eliminated from these derivations, the neutralization rule cannot.

Theorem. Every LJ+ derivation by definition 2 can be transformed into another derivation with the same endsequent and in which the cut rule does not occur.

Proof. We use Gentzen's cut elimination proof for LJ [Ge] and augment it with a few additional transformations for derivation fragments involving the neutralization rule. Transformation of derivations with cut into derivations with mix remains unchanged. The mix rule is the following:

$$\dfrac{\Gamma \mid\!\!-\, \Theta \qquad \Pi \mid\!\!-\, \Lambda}{\Gamma, \Pi^* \mid\!\!-\, \Theta^*, \Lambda}$$

Both Θ and Π contain so-called mix formula A. Θ^* and Π^* are obtained from Θ and Π by removing A from the respective formula sequences.

Apparently, formula N from the neutralization rule

$$\dfrac{N, \Gamma \mid\!\!-\, \Lambda \qquad \neg N, \Gamma \mid\!\!-\, \Lambda}{\Gamma \mid\!\!-\, \Lambda}$$

cannot be the mix formula of mix immediately below this rule. Therefore, Gentzen's proof for the case when both the left and the right rank equal one does not require any augmentation because the upper sequents of mix cannot be lower sequents of the neutralization rule.

Let us look at the part of Gentzen's proof dealing with lowering either the left or right rank. In addition to Gentzen's proof, we only have to consider two cases in which the left or right upper sequent of mix is the lower sequent of the neutralization rule. If the right upper sequent of mix is the lower sequent of the neutralization rule

$$\dfrac{\Pi \mid\!\!-\, A \qquad \dfrac{N, \Gamma \mid\!\!-\, \Lambda \qquad \neg N, \Gamma \mid\!\!-\, \Lambda}{\Gamma \mid\!\!-\, \Lambda}\ \text{neutralization}}{\Pi, \Gamma^* \mid\!\!-\, B}\ \text{mix}$$

then it is transformed into

$$\dfrac{\dfrac{\dfrac{\Pi \mid\!\!-\, A \quad N, \Gamma \mid\!\!-\, \Lambda}{\Pi, N^*, \Gamma^* \mid\!\!-\, \Lambda}\ \text{mix}}{N, \Pi, \Gamma^* \mid\!\!-\, \Lambda} \qquad \dfrac{\dfrac{\Pi \mid\!\!-\, A \quad \neg N, \Gamma \mid\!\!-\, \Lambda}{\Pi, \neg N^*, \Gamma^* \mid\!\!-\, \Lambda}\ \text{mix}}{\neg N, \Pi, \Gamma^* \mid\!\!-\, \Lambda}\ \text{thinnings and exchanges}}{\Pi, \Gamma^* \mid\!\!-\, \Lambda}\ \text{neutralization}$$

Γ^* is obtained from Γ by deleting all occurrences of A. N^* is N if A is different from N or nil otherwise. $\neg N^*$ is $\neg N$ if A is different from $\neg N$ or nil otherwise. In this transformed derivation, the rank of each of the two newly introduced mix rules is smaller than that of the original mix rule.

If the left upper sequent of mix is the lower sequent of the neutralization rule

$$\dfrac{\dfrac{N, \Gamma \mid\!\!-\, B \qquad \neg N, \Gamma \mid\!\!-\, B}{\Gamma \mid\!\!-\, B}\ \text{neutralization} \qquad \Pi \mid\!\!-\, \Lambda}{}\ \text{mix}$$

$$\Gamma, \Pi^* \mid\!-\!- \Lambda$$

then it is transformed into

$$\dfrac{\dfrac{N, \Gamma \mid\!-\!- B \qquad \Pi \mid\!-\!- \Lambda}{N, \Gamma, \Pi^* \mid\!-\!- \Lambda}\,\text{mix} \qquad \dfrac{\neg N, \Gamma \mid\!-\!- B \qquad \Pi \mid\!-\!- \Lambda}{\neg N, \Gamma, \Pi^* \mid\!-\!- \Lambda}\,\text{mix}}{\Gamma, \Pi^* \mid\!-\!- \Lambda}\,\text{neutralization}$$

Π^* is obtained from Π by deleting all occurrences of B. In this transformed derivation, the rank of each of the two newly introduced mix rules is smaller than that of the original mix rule. ♦

Seldin [Se] achieves a similar result by detouring via natural deduction and normalizing proofs there. As noted by Seldin [Se], cut elimination does not imply the sub-formula property because the neutralization rule is still retained in derivations. Therefore formulas can be discarded by application of the neutralization rule. The sub-formula property holds for predicate formulas in LJ+ though. Namely, every predicate formula in a cut-free derivation is a sub-formula of some formula in the endsequent. The proof is similar to the proof of the sub-formula property given by Gentzen in [Ge]. Just note that none of predicate formulas can be discarded by application of the neutralization rule.

The Sharpened Hauptsatz does not hold in LJ+. The following Kleene's argument [Kl] applies to LJ+ as well. Consider the following proof:

$$\dfrac{\dfrac{A(a) \mid\!-\!- A(a)}{A(a) \mid\!-\!- \exists x\, A(x)} \qquad \dfrac{A(b) \mid\!-\!- A(b)}{A(b) \mid\!-\!- \exists x\, A(x)}}{A(a) \vee A(b) \mid\!-\!- \exists x\, A(x)}$$

If the Sharpened Hauptsatz were applicable to the above LJ+ proof, then the midsequent would contain at least one occurrence of $A(a) \vee A(b)$ in the antecedent and the succedent would be $A(t)$ for some variable t. Considering all cases - $t = a$, $t = b$, t is neither a nor b - leads to a contradiction.

Derivations in LJ+ by definition 2 without cut have another interesting property. If all formulas in one occurrence of any rule are propositional, then all formulas above this rule are propositional. This follows from the fact that only propositional formulas can be discarded in derivations without cut. They are discarded by application of the neutralization rule.

Constrained Kripke Structures

Let L denote a first-order language. A Kripke structure (frame) K consists of a partially ordered set K of nodes and a function D assigning to each node k in K a domain D(k) such that if $k \le k'$, then $D(k) \subset D(k')$. In addition, a valuation T(k) maps k to a set of atomic expressions of L over D(k) for all $k \in K$. T satisfies the following: if $k \le k'$ then $T(k) \subset T(k')$.

A forcing relation is defined recursively for L as follows:

k forces atomic expression $A(d_1,\ldots,d_n)$ iff $A(d_1,\ldots,d_n) \in T(k)$
k forces A&B if k forces A and k forces B

k forces A∨B if k forces A or k forces B
k forces A⊃B if, for every such k' that k≤k', if k' forces A then k' forces B
k forces ¬A if for no such k' that k≤k', does k' force A
k forces ∀x A(x) if for every such k' that k≤k' and every d ∈ D(k'), k' forces A(d)
k forces ∃x A(x) if for some d ∈ D(k), k forces A(d)
⊤ is forced by all nodes in all Kripke structures
⊥ is not forced by any node in any Kripke structure

Any such forcing relation is known to be consistent and monotone. Intuitionistic logic is sound and complete with respect to Kripke structures [Mi, TD]
Consistency: For no sentence A and no k does k force both A and ¬A.
Monotonicity: If k ≤ k' and k forces A then k' forces A.
Soundness: If formula A is derivable in LJ, then A is forced at all nodes in every Kripke structure.
Completeness: If A is forced at all nodes in every Kripke structure, then A is derivable in LJ.

For modeling LJ+, we consider a subset of Kripke structures defined by the following constraint: if k ≤ k' and Q is a propositional symbol, then either both T(k) and T(k') contain Q or they both do not contain Q. Let us call this subset constrained Kripke structures. Our goal is to select such Kripke structures that behave like the classical truth tables on the propositional fragment of LJ+. The following theorem justifies the choice of constrained Kripke structures.

Theorem. For any propositional formula A and any node k of any constrained Kripke structure K, either k forces A or k forces ¬A. For any two nodes k and k' of K, if k ≤ k' and k' forces A, then k forces A.
Corollary. For any node k of any constrained Kripke structure K and any propositional formulas A and B, k forces A⊃B if and only if 'k forces A' implies 'k forces B', k forces ¬A if and only if k does not force A.

Proof. Both statements of this theorem are proven by induction on the size of propositional formula in question. To start proving the first statement of the theorem, consider the case that A is atomic, i.e. it is propositional symbol Q. If T(k) is contains Q, then k forces A. If T(k) does not contain A, then k forces ¬A because T(k') does not contain Q for all such k' that k≤k'. Suppose A is B&C, and the first statement holds for B and C. For any k, it sufficient to consider four cases: k forces A and B, k forces A and ¬B, k forces ¬A and B, k forces ¬A and ¬B. If k forces A and B, then it also forces A&B. In the three remaining cases, for no k' such that k≤k', does k' force both A and B. Therefore, k forces ¬(A&B). Suppose A is B⊃C, and the first statement holds for B and C. Again, it sufficient to consider four cases: k forces A and B, k forces A and ¬B, k forces ¬A and B, k forces ¬A and ¬B. If k forces A and B, or k forces ¬A and B, or k forces ¬A and ¬B, then k forces B⊃C due to the monotonicity of Kripke structures. If k forces A and ¬B, then for no k' such that k≤k', does k' force A⊃B. Therefore, k forces ¬(A⊃B). Cases of other connectives are considered similarly.

To prove the second statement of the theorem, again start from considering the case that A is atomic, i.e. it is propositional symbol Q. If k≤ k' and k' forces Q, then k forces Q by definition of the constrained Kripke structures. Suppose A is B∨C and the second statement holds for B and C. If k' forces B∨C, then k' forces B or it forces C. If k' forces B and k≤k', then k forces B by the induction assumption. Therefore, k forces B∨C as well. Considering the case that k' forces C leads to the same conclusion. Now suppose A is ¬B and the second statement holds for B. Let k

$\leq$k' and k' forces ¬B. Consider arbitrary k" such that k$\leq$k". If k" forces B, then k forces B by the induction assumption. By monotonicity, k' forces B, which contradicts the fact that k' forces ¬B. Therefore, k" does not force B and thence k forces ¬B. Cases of other connectives are considered similarly. ♦

Definition. Sequent Γ |-- Λ is called valid if for every node k in every constrained Kripke structure K, either there is such formula A from Γ that k does not force A or k forces Λ provided that Λ is not empty.

Theorem (Soundness of LJ+ with respect to constrained Kripke structures). Any sequent Γ |-- Λ derivable in LJ+ is valid.
Corollary. If |-- A, then A is forced at all nodes in every constrained Kripke structure.

Proof. Let us look at derivations in LJ+ by definition 2 without cut. We use induction on the height of derivation of Γ |-- A. Basically, we only have to check out that the neutralization rule preserves validity, that is, the lower sequent of the neutralization rule is valid whenever the upper sequents are. It is well known that all other rules that may appear in derivations preserve validity [Mi], and apparently all instances of the only LJ+ axiom schema are valid as well. Therefore, induction step can made in application to all rules except neutralization.

Let us look at the neutralization rule:

$$\frac{N, \Gamma \mid\!\text{--}\ \Lambda \qquad \neg N, \Gamma \mid\!\text{--}\ \Lambda}{\Gamma \mid\!\text{--}\ \Lambda}$$

As established earlier, for any node k of any constrained Kripke structure K, either k forces N or k forces ¬N. If k forces N, consider N, Γ |-- Λ, otherwise consider ¬N, Γ |-- Λ. By the induction assumption applied to N, Γ |-- Λ (or to ¬N, Γ |-- Λ, respectively), either there is such formula A from N, Γ (¬N, Γ) that k does not force A or k forces Λ provided that Λ is not empty. A is from Γ because it cannot be N (¬N). Therefore, Γ |-- Λ is valid. ♦

Completeness of LJ+ with respect to constrained Kripke structures will be proven later in the section Intuitionistic Nature of Median Logic. Soundness of LJ+ with respect to constrained Kripke structures is a useful tool in showing that certain formulas cannot be proved in LJ+. For instance, let us prove that ∃x P(x) ∨ ¬∃x P(x) does not hold in LJ+. Consider a constrained Kripke structure with two elements k$\leq$ m, D(k) = D(m) = { 0 }, T(k) is empty and T(m) consists of one element P(0). ∃x P(x) is not forced at k because T(k) is empty. ¬∃x P(x) is not forced at k because ∃x P(x) is forced at m and k$\leq$m. Therefore, ∃x P(x) ∨ ¬∃x P(x) is not derivable in LJ+ because k does not force it.

Intuitionistic Nature of Median Logic

Once we established the soundness of LJ+ with respect to constrained Kripke structures, we can address the issue of intuitionistic features of median logic.

Theorem. The purely predicate fragment of LJ+ is equivalent to LJ, that is, every purely predicate formula derivable in LJ+ is also derivable in LJ. There exist mixed formulas derivable in LK but not derivable in LJ+.
Corollary. The disjunction and existence properties hold in LJ+ in application to purely predicate

formulas.

Proof. Consider purely predicate formula F derivable in LJ+. Consider an arbitrary Kripke structure K (not necessarily constrained) and any node $k \in K$. Let us build another Kripke structure K' whose only difference from K is that all propositional symbols are removed from valuations T(k) for all $k \in K$. K' satisfies the definition of constrained Kripke structures simply because valuations do not have propositional symbols. Since F is purely predicate, k forces F in K if and only if k forces F in K' for any $k \in K(K')$. By soundness of LJ+ with respect to constrained Kripke structures, k forces F for all $k \in K'$. Therefore, k forces F in K as well. By completeness of LJ with respect to Kripke structures, F is derivable in LJ.

Now let us prove the second statement of this theorem. Consider formula $(\exists x\, P(x) \vee \neg\exists x\, P(x)) \vee R$ where R is a propositional symbol. This formula is provable in LK because $\exists x\, P(x) \vee \neg\exists x\, P(x)$ is. Look at the constrained Kripke structure presented earlier. It has two elements $k \le m$, $D(k) = D(m) = \{ 0 \}$, T(k) is empty and T(m) consists of one element P(0). $\exists x\, P(x)$ is not forced at k because T(k) is empty. $\neg\exists x\, P(x)$ cannot not forced at k because $\exists x\, P(x)$ is forced at m and $k \le m$. R is not forced at k either because T(k) does not contain R. Thus $(\exists x\, P(x) \vee \neg\exists x\, P(x)) \vee R$ is not forced at k. The soundness of LJ+ implies that $(\exists x\, P(x) \vee \neg\exists x\, P(x)) \vee R$ is not derivable. ♦

This theorem gives an indirect proof of admissibility of the neutralization rule in derivations of purely predicate formulas. It follows from admissibility of cut in intuitionistic logic. The neutralization rule is not admissible in case of mixed or propositional formulas though, as shown by previous examples. If the neutralization rule had been admissible, then all derivations would have been intuitionistic ones.

LJ+ coincides with LJ even on a wider class of formulas. These are the formulas in which all occurrences of every propositional symbol are of the same polarity. Given that LEM and double negation elimination are the simplest formulas having both negative and positive occurrences of propositional symbols, this class perhaps gives a reasonable syntactically-defined approximation of the boundaries within which LJ+ and LJ coincide.

Definition. A formula is called unipolar if occurrences of every propositional symbol in it are either all positive or all negative.

Theorem. Every unipolar formula derivable in LJ+ is also derivable in LJ.
Corollary. The disjunction and existence properties hold in LJ+ in application to unipolar formulas.

Proof. The proof of this theorem is by induction on the number of propositional symbols in formula F. Base case: F has no propositional symbols. The previous theorem gives a proof of the base case. Induction step: suppose this theorem holds for formulas having not more than n propositional symbols, and F has n+1 propositional symbols. Suppose R is one of F's propositional symbols, all occurrences of R are positive. The case of negative occurrences is similar to this one. If F is derivable in LJ+, so is $F\{R|\bot\}$. By the induction hypothesis, if $F\{R|\bot\}$ is derivable in LJ+, then it is derivable in LJ because it contains n propositional symbols.

LJ has the following well-known property: if $A \supset B$ is derivable and A is a positive sub-formula of F, then $F \supset F_{A|B}$ is derivable. This theorem was formulated in the chapter Median Logic Properties in application to median logic. This theorem is easily generalized on the case of multiple positive occurrences of A in F. It is done by induction on the number of the occurrences.

Since $R \supset \bot$, $F\{R|\bot\} \supset F$ is derivable in LJ and so is F due to closeness under modus ponens. ♦

Now we are equipped to prove the theorem given without proof in the section Median Logic Properties, another theorem showing boundaries of median logic, and the completeness of LJ+ with respect to constrained Kripke structures. Here is the statement of the theorem from the section Median Logic Properties again:

Theorem. If A is classically provable purely predicate formula that is not derivable in LJ+, B is strictly positive sub-formula occurrence in A, and C is a propositional formula not derivable in LJ+, then $A_{B|B \vee C}$ does not hold in LJ+. If D is any propositional formula, then $A_{B|B\&D}$ does not hold in LJ+.

Proof. Since A is not derivable in LJ, there is Kripke structure K and node k in it such that k does not force A due to completeness of LJ with respect to Kripke structures. It is safe to assume that K does not contain propositional symbols (see the proof of the previous theorem). K is a constrained Kripke structure. If k forces B, then k may or may not force B&D. Given the rules for calculating forcing relationship for connectives & and $\vee$, k still does not force $A_{B|B\&D}$ in either case. If k does not force B, then it does not force B&D either. Again by forcing definition for & and $\vee$, k does not force $A_{B|B\&D}$. Therefore, $A_{B|B\&D}$ is not derivable in LJ+ because of soundness of LJ+ with respect to constrained Kripke structures.

Now consider the case with disjunction. Let us pick a classical logic model in which C is false. Extend K by adding elements to all D(k) and T(k) in accordance with the selected classical model, i.e. propositional symbol R is added to D(k) and T(k) for all $k \in K$ if and only if R is true in the selected classical model. This extended structure K' is apparently constrained. If k forces B in K', then k forces $B \vee C$ too. If k does not force B, then k does not force $B \vee C$ in K' because C is not forced by any node from K'. By forcing definition for & and $\vee$, k does not force $A_{B|B \vee C}$. This formula is not derivable in LJ+ because of soundness of LJ+ with respect to constrained Kripke structures. ♦

This theorem assures that variants of LEM for purely predicate formulas do not hold in median logic. For example, if R is a propositional symbol, then neither $(\exists x\, P(x) \vee R) \vee \neg\exists x\, P(x)$ nor $(\exists x\, P(x)\ \&\ (R \supset R)) \vee \neg\exists x\, P(x)$ holds in LJ+. Likewise, formulas combining double negation elimination for purely predicate formulas with propositional symbols do not hold in median logic either. The following theorem gives a hint for constructing such formulas.

Theorem. If $A \supset B$ is a classically derivable purely predicate formula that is not derivable in LJ+, C is a propositional formula derivable in LJ+, D is a propositional formula not derivable in LJ+, then neither $A\&C \supset B$ nor $A \supset (B \vee D)$ holds in LJ+.

Proof. Since $A \supset B$ is not derivable in LJ, there is Kripke structure K and such node k in it that k does not force $A \supset B$ due to completeness of LJ with respect to Kripke structures. Let us select such $k \in K$ that k forces A and does not force B. As above, it is safe to assume that K does not contain propositional symbols, and thus, K is a constrained Kripke structure. Since C is derivable in LJ+, k forces A&C as well. Therefore, k does not force $A\&C \supset B$ and hence $A\&C \supset B$ is not derivable in LJ+ because of its soundness with respect to constrained Kripke structures. Using the same argument as in the previous theorem, we can extend K to another constrained Kripke structure in such a way that k does not force $B \vee D$. Therefore, k does not force $A \supset (B \vee D)$ and it is not derivable in LJ+. ♦

Here are a couple of sample variants of double negation elimination: $(\neg\neg(\exists x\, P(x))\ \&\ (A \supset A \vee B)) \supset \exists x\, P(x)$ and $\neg\neg\exists x\, P(x) \supset (\exists x\, P(x) \vee A)$. These formulas are not derivable in LJ+ as guaranteed by the last theorem.

Theorem (Completeness of LJ+ with respect to constrained Kripke structures). If formula F is forced at any node k in any constrained Kripke structure K, then F is derivable in LJ+.

Proof. The proof of this theorem is by induction on the number of propositional symbols in F. Base case: F has no propositional symbols. Consider an arbitrary Kripke structure K (not necessarily constrained) and any node $k \in K$. As explained earlier, it is safe to assume that K does not contain propositional symbols. Therefore, K is a constrained structure and k forces F. By the completeness of LJ with respect to Kripke structures, F is derivable in LJ and hence in LJ+.

Induction step: suppose this theorem holds for formulas having not more than n propositional symbols, and F has n+1 propositional symbols. If R is one of F's propositional symbols, then F is derivable in LJ+ if both $F\{R|\top\}$ and $F\{R|\bot\}$ are derivable in LJ+ (by the second theorem from the section Classical Nature of Median Logic.) Consider an arbitrary constrained Kripke structure K not containing R in any domain D, and consider any node $k \in K$. Since k forces F, k also forces $F\{R|\bot\}$ because forcing is identical for k paired up with any atomic sub-formula of F and k paired up with the respective atomic sub-formula of $F\{R|\bot\}$, and the structure of both formulas is the same. Now extend K with R belonging to all D(k) and all T(k). Since k forces F, k also forces $F\{R|\top\}$ by the same argument. Apparently, presence of R in D(k) and T(k) does not affect forcing for $F\{R|\top\}$ and $F\{R|\bot\}$ because they do not contain R. Therefore, $F\{R|\top\}$ and $F\{R|\bot\}$ are forced by any k in any constrained Kripke structure, and thus they are derivable in LJ+ by the induction hypothesis, and so is F. ♦

Disjunction and Existence Properties

As noted earlier, the disjunction and existence properties are perhaps the two most important features of intuitionistic logic. Once the boundaries of LJ within LJ+ have been characterized, it is clear that both properties hold for the formulas within the LJ boundaries. Apparently, the disjunction property does not hold for the entire median logic because it does not hold for classical propositional logic. Median logic was defined with an opposite goal in mind for its propositional fragment. The existence property does not hold in LJ+ either. Another question is: What is the scope of LJ+ outside LJ in which the disjunction and existence properties hold? The following theorems give an insight on the extent of the disjunction and existence properties.

Theorem. If formula $F \vee G$ is derivable in LJ+, F and G do not have common propositional symbols, then either F or G is derivable in LJ+.

Proof. This proof is based on 'the gluing method' suggested by Kurokawa [Ku]. It is similar to Kurokawa's but applies to a different framework and yields a somewhat stronger result. Let F' denote the set of propositional symbols in F and G' denote the set of propositional symbols in G. If neither F nor G is derivable in LJ+, then there are such constrained Kripke structures K and M that F is not forced at some node $k' \in K$ and G is not forced at some node $m' \in M$. Let us 'glue' these two structures by adding a new node n such that $n \le k$ for all $k \in K$, $k' \le k$ and $n \le m$ for all $m \in M$, $m' \le m$. No non-propositional atomic expression $A(d_1,\ldots,d_n)$ is forced at n. Let the set of propositional symbols forced at n is the union of the propositional symbols forced at k' and the propositional symbols forced at m'. Also, let us add these propositional symbols forced at m' to the set of forced atomic expressions for every node in K, and let us add the propositional symbols forced at k' to the set of forced atomic expressions for every node in M.

This newly built combination of n, K and M is a constrained Kripke structure. Note that no pair $<k,m>$ where $k \in K$ and $m \in M$ is ordered; $k \leq n$, $m \leq n$ do not hold either. Adding propositional symbols not occurring in F to the set of forced atomic expressions for all nodes of K does not affect valuation monotonicity. Neither this addition changes forcing relationships for nodes from K. The same is true about G and M, respectively. Similarly, all propositional symbols forced at n are also forced at other nodes $k \in K$, $k' \leq k$ and $m \in M$, $m' \leq m$. Therefore, we defined a Kripke structure. This Kripke structure is constrained because any propositional symbol forced at any $k \in K$, $n \leq k$ is also forced at n and every propositional symbols forced at any $m \in M$, $n \leq m$ is forced at n as well.

Neither formula F nor formula G is forced at n in the glued constrained Kripke structure, and hence, formula $F \vee G$ is not forced at n either. Therefore either F or G should be derivable.♦

The existence property is more interesting when logic formulations include functional symbols and object constants. In our simplistic formulation without functional symbols and even without object constants, the existence property is reduced to a statement about a variable as opposed to a term.

We show that the existence property does not hold in LJ+ by constructing a counterexample. Formula $\exists x\ (A(a) \vee N)\&(A(b) \vee \neg N)) \supset A(x)$ is derivable in LJ+ if N is a propositional formula. Here is how it is derived.

```
N |-- N                     A(a) |-- A(a)
========                    =========
N, ¬N |-- A(a)              A(a), ¬N |-- A(a)
------------------------------------------------
            A(a)∨N, ¬N |-- A(a)
```

Formula $A(b) \vee \neg N, N$ |-- $A(b)$ is derived in a similar manner. Now, continue the above derivation.

```
(A(a)∨N)&(A(b)∨¬N), ¬N |-- A(a)
------------------------------------------
¬N |- ((A(a)∨N)&(A(b)∨¬N))⊃A(a)
-------------------------------------------
¬N |- ∃x ((A(a)∨N)&(A(b)∨¬N))⊃A(x)
```

Similarly, one can derive N |- $\exists x\ ((A(a) \vee N)\&(A(b) \vee \neg N)) \supset A(x)$ from $A(b) \vee \neg N, N$ |-- $A(b)$. Finally, application of the neutralization rule gives the formula in question.

In order to show that the existence property does not hold for this formula, we have to consider all possibilities for the variable replacing x: it is a; it is b; it is neither a nor b. Thus, it is sufficient to show that none of the three formulas below is derivable in LJ+:
$(A(a) \vee N)\&(A(b) \vee \neg N)) \supset A(a)$
$(A(a) \vee N)\&(A(b) \vee \neg N)) \supset A(b)$
$(A(a) \vee N)\&(A(b) \vee \neg N)) \supset A(c)$
None of these is even classically derivable. It can be shown by building classical logic

interpretations in which these formulas are false. Let the domain of these interpretations be { 0, 1, 2 }. In the interpretation applied to the first and third formula, the set of true atomic values is comprised of N and A(0). Only A(0) is true in the interpretation applied to the second formula. The first formula evaluates to false when a=1, b = 0. The second formula evaluates to false when a=0, b=1. The third formula evaluates to false when a=1, b=0, c=2.

Theorem. The existence property holds for the following three classes of formulas derivable in LJ+:

1. ∃x (A(x)&B) if A(x) is unipolar
2. ∃x (A(x)⋁B) if A(x) is unipolar, A(X) and B do not share propositional symbols
3. ∃x (B⊃A(x)) if A(x) is unipolar, B is purely propositional, A(X) and B do not share propositional symbols

Proof. Formula ∃x (A(x)&B) is equivalent to ∃x A(x)&B in classical, intuitionistic and median logic. Similarly, formula ∃x (A(x)⋁B) is equivalent to ∃x A(x) ⋁ B. If ∃x A(x) & B is derivable in LJ+, then both ∃x A(x) and B ought to be derivable in LJ+ due to closeness under modus ponens. Since ∃x A(x) is derivable in LJ as well, A(t) is intuitionistically derivable for some variable t. Therefore, A(t)&B is derivable. A similar reasoning applies to the second class. If ∃x A(x) ⋁ B is deriavable in LJ+, then either ∃x A(x) or B is derivable by the above theorem about the disjunction property. If B is derivable, then A(t)⋁B is derivable for any t. If ∃x A(x) is derivable, it is intuitionistically derivable as well. Consequently, A(t) is derivable for some variable t, and therefrom A(t)⋁B is derivable. As proved earlier, B⊃A(x) is equivalent to ¬B⋁A(x) in LJ+ provided that B is propositional. Therefore, ∃x (¬B⋁A(x)) is derivable in LJ+. Now, the reasoning for the second class can be applied to this case too. Finally, derivability of ¬B⋁A(t) implies derivability of B⊃A(t). ♦

Other Formulations

We outline two other formulations of median logic. One is for natural deduction and the other is for Hilbert-style systems. These formulations will be based on Gentzen's definitions of NJ and LHJ, respectively. Refer to [Ge] or [Pr] for a complete definition of NJ, and to [Ge] or [Kl] for a complete definition of LHJ.

Definition. NJ+ is NJ plus additional axiom schema A⋁¬A that holds for propositional formulas only.

Definition. LHJ+ is LHJ plus additional axiom schema A⋁¬A that holds for propositional formulas only.

Theorem. LHJ+, NJ+, LJ+ are equivalent, that is, any formula derivable in one of them is also derivable in the other two.

Proof. Gentzen proved that LJ, NJ, and LHJ are equivalent [Ge]. In order to prove equivalence of LJ+, NJ+, and LHJ+, we will follow Gentzen's proof of equivalence of LK, NK, and LHK [Ge]. First, let us transform a LHJ+ derivation to into an equivalent NJ+ derivation. This transformation is done as in Gentzen's transformation of LHJ derivations into equivalent NJ derivations. Additional basic formulas according to the schema A⋁¬A (for propositional formulas) remain unchanged. They become basic formulas of NJ+ derivations. Similarly for transformation of NJ+

derivations into equivalent LJ+ derivations, we proceed as in Gentzen's transformation of NJ derivations into equivalent LJ ones. Additional basic formulas according to the schema A∨¬A (for propositional formulas) are transformed into the respective basic sequents in LJ+ by definition 1. Finally, transformation of LJ+ derivations by definition 1 into equivalent LHJ+ derivations is also done by performing Gentzen's transformation of LJ derivations into equivalent LHJ derivations. The additional basic sequents A∨¬A with propositional formula A are transformed into the respective basic formulas in NJ+. ♦

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