Some remarks on Approximation Properties with Applications

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Abstract. We study some known approximation properties and introduce and investigate several new approximation properties, closely connected with different quasi-normed tensor products. These are the properties like the AP_s or $AP_{(s,w)}$ for $s \in (0,1]$, which give us the possibility to identify the spaces of s-nuclear and (s,w)-nuclear operators with the corresponding tensor products (e.g., related to Lorentz sequence spaces). Some applications are given (in particular, we present not difficult proofs of the trace-formulas of Grothendieck-Lidskii type for several ideals of nuclear operators).

Our main reference is [10]. All the notions , notations and facts, we use without any reference, can be found in [1, 2, 4, 8, 10, 12].

I. The Grothendieck approximation property for a Banach space X can be defined as follows: X has the AP iff for every sequence $(x_n)_{n=1}^\infty \subset X$ tending to zero, for any $\varepsilon > 0$ there exists a finite rank (continuous) operator R in X such that for each $n \in \mathbb{N}$ one has $||Rx_n - x_n|| \le \varepsilon$. Consider a natural question: for which sequences $(x_n) \in c_0(X)$, under some additional assumptions, the identity map id_X surely can be approximated by finite rank operators, as above, and which of those conditions are sharp (or, if one wishes, optimal)?

One of the simplest fact (we think, known for more than 30 years) that

$(*)$ if $(x_n) \in l_2(X)$, X is any, then the answer is positive.

Here ia a reason of this: Assuming $||x_n|| \searrow 0$, take any $N \in \mathbb{N}$ and consider the linear span $\mathrm{span}[x_n]_1^N =: E_N$ as a subspace of X. Define, fixing an $\varepsilon > 0$, a finite rank R to be a projection from X onto E_N whose norm $\le \sqrt{N}$.

Now if N is such that, for every $n \ge N$, we have $||x_n|| \le \frac{\varepsilon}{\sqrt{N}+1}$, then

$$||Rx_n - x_n|| = 0 \ \text{ if } \ n \le N,$$

and

$$||Rx_n - x_n|| \le \varepsilon.$$

Of course, instead of $(*)$ we can consider the the statement

$(**)$ if $(x_n) \in l^0_{2,\infty}(X)$ [Lorentz space with "o" small], X is any, then the answer is positive.

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The idea of the above proof is very simple and can be applied in some more general situations. For instance, every subspace of finite dimension of an L_p-space is $n^{|1/2-1/p|}$-complemented in that L_p-space. So, if $p \in [1, \infty]$, $\alpha = |\frac{1}{2} - \frac{1}{p}|$ and X is a subspace of an L_p-space, then

$(***)$ for every sequence $(x_n) \in l^0_{q,\infty}(X)$, where $1/q = \alpha$, the answer is positive.

Remark 1: About sharpness: it will be discussed a little bit later.

Remark 2: The statement $(***)$ has, as a matter of fact, the following quantitative aspect: Given $\alpha \in [0, 1/2]$ and a Banach space X with the property that every finite dimensional subspace F of X is contained in a finite dimensional subspace $E \subset X$, which (E) in turn is $C\,(\dim F)^\alpha$-complemented in X, we have

$(***)'$ for every sequence $(x_n) \in l^0_{q,\infty}(X)$, where $1/q = \alpha$, for any $\varepsilon > 0$ there is a finite rank operator R in X so that $\sup_n ||Rx_n - x_n|| \le \varepsilon$.

Particular cases:

(i) $q - 2$ and $\alpha = 1/2$ or $q = \infty$ and $\alpha = 0$ $(=$ "X is any Banach space" or "X is isomorphic to a Hilbert space");

(ii) $(x_n) \in l_q(X), q \in [2, \infty)$, or $(x_n) \in c_0(X), q = \infty$ [Hilbert case].

For a while let us introduce the notions of the corresponding approximation properties for a Banach space X (taking into account that the possibility of approximations on c_0-sequences by finite rank operators gives us the Grothendieck's approximation property AP) : Let $0 < q \le \infty$ and $1/s = 1/q + 1$. We say that X has the $\widetilde{AP}_s$ [resp., the $\widetilde{AP}_{s,\infty}$] if for every $(x_n) \in l_q(X)$ [resp., $l^0_{q,\infty}$] (where $l_q(X)$ means $c_0(X)$ for $q = \infty$) and for every $\varepsilon > 0$ there exists a finite rank operator $R \in X^* \otimes X$ such that $\sup_n ||Rx_n - x_n|| \le \varepsilon$. Trivially, e.g., $\widetilde{AP}_{s_2} \implies \widetilde{AP}_{s_1}$ if $s_1 \le s_2$. Thus, $\widetilde{AP}_1 (= AP)$ implies any $\widetilde{AP}_s$.

The statement $(*)$ (and $(**)$) says that every Banach space has the above property $\widetilde{AP}_{2/3}$ (and even the $\widetilde{AP}_{2/3,\infty}$). The statement $(***)$ gives the corresponding result for L_p-subspaces. Moreover, the assertion mentioned in Remark 2, shows that, for instance, any subspace of any quotient (= any quotient of any subspace) of a Banach space of type 2 (resp., of cotype 2) and of cotype p, $p \in [2, \infty)$ (resp., of type p'), possesses the $\widetilde{AP}_s$ (even the $\widetilde{AP}_{s,\infty}$) with $1/s = 1 + |1/2 - 1/p|$.

II. Let us recall that the notion of the AP of Grothendieck can be reformulated in terms of the projective tensor products "$\widehat{\otimes}$". Namely, a Banach space X has the AP iff for every Banach space Y the canonical (natural) mapping $Y^* \widehat{\otimes} X \to L(Y, X)$ is one-to-one (or, what is the same, the natural mapping $X^* \widehat{\otimes} X \to L(X) := L(X, X)$ is injective). In [3], A. Grothendieck has considered also some other tensor products (linear subspaces of "$\widehat{\otimes}$"'s), which we will denote by "$\widehat{\otimes}_s$" for $0 < s \le 1$ (so that $\widehat{\otimes} = \widehat{\otimes}_1$) : For Banach spaces X and Y, let $Y^* \widehat{\otimes}_s X$ be a subspace of the projective tensor product $Y^* \widehat{\otimes} X$ consisting of the tensors $z \in Y^* \widehat{\otimes} X$, which admit representations of the form

$$z = \sum_{n=1}^{\infty} \lambda_n y'_n \otimes x_n, \tag{1}$$

where $(\lambda_n) \in l_s$, (y'_n) and (x_n) are bounded sequences from Y^* and X respectively. With a natural "quasi-norm" (see [10]) the linear subspace $Y^*\widehat{\otimes}_s X$ of the space $Y^*\widehat{\otimes} X$ can be considered as a "quasi-normed tensor product" (it is then a complete metric space [3]).

One of the nice (with a non trivial proof in [3]) theorem of Grothendieck is the fact that the natural map from $Y^*\widehat{\otimes}_{2/3} X$ into $L(Y,X)$ is injective for any Banach spaces X, Y. Let us compare this Grothendieck's result with a simple assumption in Section I, where "$s = 2/3$" was appeared. Must be clear that it is not a chance coincidence, and really we have

Theorem 2.1. For $s \in (0,1]$ and for a Banach space X the following are equivalent:

1) X has the $\widetilde{AP}_s$ in the sense of the definition in Section I;

2) X has the AP_s in the sense of the definition in [13], i.e. for every Banach space Y the natural mapping $Y^*\widehat{\otimes}_s X \to L(Y,X)$ is one-to-one.

Let us mention also that

(AP_s) A Banach space X has the AP_s, $0 < s \leq 1$, iff the canonical map $X^*\widehat{\otimes}_s X \to L(X)$ is one-to-one (or, what is the same, there exists no tensor element $z \in X^*\widehat{\otimes}_s X$ with trace $z = 1$ and $\widetilde{z} = 0$, where $\widetilde{z}$ is the associated (with z) operator from X to X).

The analogous theorems and facts are maybe valid for the $\widetilde{AP}_{s,\infty}$ and the $AP_{s,\infty}$ from [13] (see a small discussion below).

Proof of the assertion (AP_s). Suppose X has the AP_s, but there exists a Banach space Y such that the natural map $Y^*\widehat{\otimes}_s X \to L(Y,X)$ is not one-to-one.Take an element $z \in Y^*\widehat{\otimes}_s X$ which is not zero, but generates a zero operator $\widetilde{z}: Y \to X$. Then we can find an operator $U \in L(X, Y^{**})$ so that trace $U \circ z = 1$. If $z = \sum_{k=1}^\infty \lambda_k\, y'_k \otimes x_k$ is a representation of z in $Y^*\widehat{\otimes}_s X$ $((\lambda_k) \in l_s, (x_k)$ and (y'_k) are bounded), then

$$1 = \text{trace } z = \sum_{k=1}^\infty \lambda_k \langle Ux_k, y'_k\rangle = \sum_{k=1}^\infty \lambda_k\, \langle x_k, U^* y'_k\rangle$$

and $\sum_{k=1}^\infty \lambda_k\, U^* y'_k(x) x_k = 0$ for every $x \in X$. Put $x'_k := U^* y'_k$, $z_0 := \sum_{k=1}^\infty x'_k \otimes x_k \in X^*\widehat{\otimes}_s X$. We have

$$\text{trace } z_0 = 1,\ \widetilde{z}_0 \neq 0$$

(by assumption on X). Consider a 1-dimensional operator $R = x' \otimes x$ in X with the property that trace $R \circ z_0 > 0$. Then

$$0 < \text{trace } R \circ z_0 = \sum_{k=1}^\infty \lambda_k\, \langle x'_k, x\rangle \langle x', x_k >= \sum_{k=1}^\infty \langle U^* y'_k, x\rangle \langle x', x_k\rangle$$

$$= \langle \sum_{k=1}^\infty \lambda_k\, \langle Ux, y'_k\rangle x_k, x\rangle = \langle x', \sum_{k=1}^\infty \lambda_k\, U^* y'_k(x) x_k\rangle = 0.$$

Proof of Theorem 2.1. We will use the assertion (AP_s).

1) $\implies$ 2). Let $z \in X^*\widehat{\otimes}_s X$ and trace $z=1$. Write $z=\sum \lambda_k\, x'_k\otimes x_k$, where the sequences $x'_k)$ and (x_k) are bounded and $(\lambda_k)\in l_s$, $\lambda_k\ge 0$, (λ_k) is non-increasing. Then

$$z=\sum_{k=1}^{\infty}(\lambda_k^s\, x'_k)\otimes(\lambda_k^{1-s})$$

(recall that $1/s=1+1/q$; so $1-1/s=1/q$). The sequence $(\lambda_k^{1-s}x_k)$ is in $l_q(X)$. By 1), for every $\varepsilon>0$ there exists a finite rank operator $R\in X^*\otimes X$ such that $||R(\lambda_k^{1-s}x_k)-\lambda_k^{1-s}||\le\varepsilon$ for each $k\in\mathbb{N}$. It follows that, for this operator R,

$$|\operatorname{trace}(z-R\circ z)|=|\sum_{k=1}^{\infty}\langle\lambda_k^s x'_k,\lambda_k^{1-s}x_k-R(\lambda_k^{1-s}x_k)\rangle|\le\sum_{k=1}^{\infty}\lambda_k^s||x'_k||\cdot\varepsilon\le const\cdot\varepsilon.$$

Hence, for small $\varepsilon>0$ we have that, for an operator $R\in X^*\otimes X$,

$$|\operatorname{trace} R\circ z|\ge 1/2$$

and therefore z generates a non-zero operator $\widetilde{z}$.

Before consider a proof of the implication 2) $\implies$ 1) we will make some additional remarks. We collect the remarks in

Lemma 2.1. Let $s\in(0,1]$, $q\in(0,\infty]$, $1/s=1+1/q$. For $a:=(a_k)\in l_1$ and $b:=(b_k)\in l_q$ we have

$$(\sum_{k=1}^{\infty}|a_kb_k|^s)^{1/s}\le\sum_{k=1}^{\infty}|a_k|\cdot(\sum_{k=1}^{\infty}|b_k|^q)^{1/q}. \tag{2}$$

Moreover,

$$||a||_{l_1}=\sup_{||b||_{l_q}=1}(\sum_{k=1}^{\infty}|a_kb_k|^s)^{1/s}$$

(if $q=\infty$, the evident changes have to be made in (2)).

Proof of Lemma 2.1. We may consider the case where $q\in(0,\infty)$. Putting $p:=1/s$ (then $1/p'=1-s=s/q$ and $sp'=q$), we obtain

$$\sum_{k=1}^{\infty}|a_kb_k|^s\le(\sum_{k=1}^{\infty}|a_k|^{sp})^{1/p}\cdot(\sum_{k=1}^{\infty}|b_k|^{sp'})^{1/p'}=(\sum_{k=1}^{\infty}|a_k|)^s\cdot(\sum_{k=1}^{\infty}|b_k|^q)^{s/q}.$$

For the second part: Let $a=(a_k)\in l_1$. Take $b_k:=\frac{|a_k|^{1/q}}{||a||_{l_1}^{1/q}}$. Then $\sum_{k=1}^{\infty}|b_k|^q=\sum_{k=1}^{\infty}\frac{|a_k|}{||a||_{l_1}}=1$ and

$$(\sum_{k=1}^{\infty}|a_kb_k|^s)^{1/s}==(\sum_{k=1}^{\infty}\frac{|a_k|^{s/q}}{||a||_{l_1}^{s/q}}|a_k|^s)^{1/s}=(\sum_{k=1}^{\infty}\frac{|a_k|^{s/q+s}}{||a||_{l_1}^{s/q}})^{1/s}$$

$$=(\sum_{k=1}^{\infty}\frac{|a_k|^{s(1+1/q)}}{||a||_{l_1}^{s/q}})^{1/s}=(\sum_{k=1}^{\infty}\frac{|a_k|}{||a||_{l_1}^{s/q}})^{1/s}=\frac{(\sum_{k=1}^{\infty}|a_k|)^{1/s}}{||a||_{l_1}^{1/q}}=(/\sum_{k=1}^{\infty}|a_k|)^{1/s-1/q}=||a||_{l_1}.$$

Proof of Theorem 2.1 (continuation).

2) $\implies$ 1). Let X has the AP_s, but does not have the $\widetilde{AP_s}$, $1/s = 1 + 1/q$. Then there is a sequence $(x_n) \in l_q(X)$ (if $q = \infty$, we consider a sequence from $c_0(X) = l^0_\infty(X)$) such that there exists an $\varepsilon > 0$ with the property that for any finite rank operator $R \in X^* \otimes X$ one has $\sup_n ||Rx_n - x_n|| > \varepsilon$. Consider the space $C_0(K; X)$ for $K := \{x_n\}_{n=1}^\infty \cup \{0\}$. Every operator U in X can be considered as a continuous function on K with values in X by setting $f_U(k) := U(k)$ for $k \in K$. In particular, for the identity map id in X and for any $R \in X^* \otimes X$ we have

$$||f_{\mathrm{id}} - f_R||_{C_0(K;X)} \geq \varepsilon.$$

The subset $\mathcal{R} := \overline{\{f_R :\ R \in X^* \otimes X\}}^{C_0(K;X)}$ of $C_0(K; X)$ is a closed linear subspace in $C_0(K; X)$. So, there exists an X^*-valued measure $\mu = (x'_k)_{k=1}^\infty \in C_0^*(K; X) = l_1(\{x_n\}_{n=1}^\infty) \cup \{0\}; X)$ such that $\mu|_{\mathcal{R}} = 0$ and $\mu(f_{id}) = 1$. In other words, we can find a sequence (x'_k) with $\sum_{k=1}^\infty ||x'_k|| < \infty$ such that $\sum_{k=1}^\infty \langle x'_k, x_k\rangle = 1$ and $\sum_{k=1}^\infty \langle x'_k, Rx_k\rangle = 0$ for any $R \in X^* \otimes X$.

Define a tensor element $z \in X^* \widehat{\otimes} X$ by $z := \sum_{k=1}^\infty x'_k \otimes x_k$. Since $(x_k) \in l_q(X)$ and $(x'_k) \in l_1(X^*)$, we get from Lemma 2.1 that

$$\Big(\sum_{k=1}^\infty ||x'_k||^s\, ||x_k||^s\Big)^{1/s} \leq \sum_{k=1}^\infty ||x'_k|| \cdot \Big(\sum_{k=1}^\infty ||x_k||^q\Big)^{1/q}.$$

Therefore, $z \in X^* \widehat{\otimes}_s X$, trace $z = \sum_{k=1}^\infty \langle x'_k, x_k\rangle = 1$ and trace $R \circ z = 0$ for every $R \in X^* \otimes X$. This means that X does not have the AP_s.

After Theorem 2.1 is proved, we can make a conclusion: $AP_s = \widetilde{AP}_s$ for any $s \in (0, 1]$.

III. Now we are going to discuss some questions around the properties $\widetilde{AP}_{s,\infty}$ and $AP_{s,\infty}$. The $\widetilde{AP}_{s,\infty}$ was defined above. Recall the definition of the $AP_{s,\infty}$ from, e.g., [13]: We say that a Banach space X has the $AP_{s,\infty}$, $0 < s < 1$, if for every Banach space Y the natural mapping $Y^* \widehat{\otimes}_{s\infty} X \to L(Y, X)$ is one-to-one, where

$$Y^* \widehat{\otimes}_{s\infty} X = \{z \in Y^* \widehat{\otimes} X :\ z = \sum_{k=1}^\infty \lambda_k y'_k \otimes x_k,\ (x_k) \text{ and } (y'_k) \text{ are bounded, } (\lambda_k) \in l^0_{s\infty}\}.$$

Let us consider the connections between the $AP_{s,\infty}$ and the $\widetilde{AP}_{s,\infty}$. For a partial discussion of this we need a lemma, which follows from Lemma 2.1 by interpolation in Lorentz spaces.

Lemma 3.1. Let $s \in (0, 1), q \in (0, \infty), 1/s = 1+1/q, r \in (0, \infty]$. If $a = (a_k) \in l_1$, $b = (b_k) \in l_{qr}$, then $ab := (a_k b_k)_{k=1}^\infty \in l_{sr}$. In particular, for $a \in l_1$ and $b \in l_{q\infty}$ the sequence ab is in $l_{s\infty}$ (thus, evidently, in $l^0_{s\infty}$).

Proof of Lemma 3.1 consist of the applications of Lemma 2.1 and the general interpolation theorem for the multiplication operator $\widetilde{a}$, induced by a fixed sequence $a = (a_k) \in l_1 : \widetilde{a}$ maps (b_k) to $(a_k b_k)$.

Namely, fix $s \in (0,1), q \in (0,\infty)$ with $1/s = 1 + 1/q$. Take $s_1, s_2 \in (0,1)$ and $q_1, q_2 \in (0,\infty)$ so that for some $\theta \in (0,1)$ we have

$$\frac{1}{q} = (1-\theta)\frac{1}{q_1} + \frac{1}{q_2},\ 0 < \frac{1}{s_2} < \frac{1}{s} < \frac{1}{s_1} < \infty,\ 0 < \frac{1}{q_2} < \frac{1}{q} < \frac{1}{q_1} < \infty,$$

and

$$\frac{1}{s_1} = 1 + \frac{1}{q_1},\ \frac{1}{s_2} = 1 + \frac{1}{q_2}.$$

By Lemma 2.1, $\widetilde{a}$ maps $l_{q_1 q_1}$ into $l_{s_1 s_1}$ and $\widetilde{a}$ maps $l_{q_2 q_2}$ into $l_{s_2 s_2}$. Applying, e.g., Theorem 5.3.1 from [1] or other results from the pages 113-114 in [1], we get that $\widetilde{a}$ maps l_{qr} into l_{sr}, $0 < r \le \infty$ (note that $1/s = 1 + 1/q = 1 + (1-\theta)/q_1 + \theta/q_2 = (1-\theta) + \theta + (1-\theta)/q_1 + \theta/q_2 = (1-\theta)(1+1/q_1) + \theta(1+1/q_2) = (1-\theta)/s_1 + \theta/s_2$).

Remark 3.1: As a matter of fact, $l_1 \cdot l_{q\infty} = l_{s1}$ in Lemma 3.1. We need now only the above inclusion.

Now let $t \in (0,1]$, $p \in (0,\infty]$, $r \in (0,\infty]$ and consider a tensor product $\widehat{\otimes}_{t;p,r}$, defined in the following way: For a couple of Banach spaces X, Y the tensor product $Y^*\widehat{\otimes}_{t;p,r}X$ consists of those elements z of the projective tensor product $Y^*\widehat{\otimes}X$ which admit representations of the type

$$z = \sum_{k=1}^{\infty} a_k b_k\, y_k' \otimes x_k;\ (y_k') \text{ and } (x_k) \text{ are bounded, } (a_k) \in l_t,\ (b_k) \in l_{pr}$$

(recall that everywhere here we consider $l^0_{p\infty}$ in the case $r = \infty$).

Remark 3.2: As was noted in Remark 3.1, $l_1 \cdot l_{q\infty} = l_{s1} (\subset l^0_{s\infty} \subset l_{s\infty})$, where $0 < s < 1, 1/s = 1 + 1/q$, We have also

$$l_{s1} = l_1 \cdot l^0_{q\infty} \text{ and } l_1 \cdot l_{q\infty} = l_1 \cdot l^0_{q\infty}$$

(so, for example, in the definition of $\widehat{\otimes}_{1;q,\infty}$ one can assume that $(a_k) \in l_1$ and $(b_k) \in l^0_{q\infty}$). Indeed, if we use the equality $l_1 \cdot l_{q\infty} = l_{s1}$, take $d \in l_{s1}$ (assuming $d = d^* = (d_k^*)$). Then $\sum_{k=1}^{\infty} k^{1/s}\, d_k^*/k < \infty$, i.e. $\sum_{k=1}^{\infty} k^{1/q}\, d_k^* < \infty$. Let $\varepsilon = (\varepsilon_k)$ be a scalar sequence such that $\varepsilon_k \searrow 0$ and $\sum_{k=1}^{\infty} \varepsilon_k^{-1} d_k^* k^{-1/q} < \infty$. Put

$$\alpha_k := \frac{d_k^* k^{1/q}}{\varepsilon_k,}\ \beta_k := \frac{\varepsilon_k}{k^{1/q}}.$$

Then $\alpha := (\alpha_k) \in l_1$ and $\beta := (\beta_k) \in l^0_{q\infty}$. So, $d = \alpha\beta \in l_1 \cdot l^0_{q\infty}$. Another way (not to use "l_{s1}"): Let $0 < q < \infty$, $\alpha \in l_1$, $\beta \in l_{q\infty}$ (assuming, without loss of generality, that $\beta = \beta^*$). Consider a sequence $\varepsilon := (\varepsilon_k)$ such that $\varepsilon_k \searrow 0$ and $(\alpha_k/\varepsilon_k) \in l_1$. Put $\widetilde{\alpha} := \alpha/\varepsilon = (\alpha_k/\varepsilon_k)$ and $\widetilde{beta} := \varepsilon\beta = (\varepsilon_k \beta_k)$. Then $\widetilde{\alpha} \in l_1$, $\widetilde{\beta} \in l^0_{q\infty}$ and $\alpha\beta = \widetilde{\alpha}\widetilde{\beta} \in l_1 \cdot l^0_{q\infty}$.

Let us say that X has the $AP_{t;p,r}$, if for every Banach space Y and for t, p, r as above the canonical mapping $Y^*\widehat{\otimes}_{t;p,r}X \to L(Y,X)$ is one-to-one.

By Lemma 3.1, if $s \in (0,1)$ and $1/s = 1 + 1/q$, then $\widehat{\otimes}_{1;q,\infty} \subset \widehat{\otimes}_{s,\infty}$. Therefore, we get

Corrolary 3.1. If $s \in (0,1)$ and $1/s = 1 + 1/q$, then $AP_{s,\infty} \implies AP_{1;q,\infty}$.

Evidently, also $AP_{s,\infty} \implies AP_s$ (for $s \in (0,1)$).

Theorem 3.2. Let $s \in (0,1), q \in (0,\infty)$ and $1/s = 1+1/q$. If X has the $AP_{1;q,\infty}$, then X has the $\widetilde{AP}_{s,\infty}$. In particular, $AP_{s,\infty} \implies \widetilde{AP}_{s,\infty}$.

Proof. It is enough to repeat word for word the proof of the implication 2) $\implies$ 1) of Theorem 2.1 ("continuation"), just changing "$l_q(X)$" by "$l^0_{q,\infty}$" (no necessity to apply Lemma 2.1 or Lemma 3.1).

Remark 3.3. In this moment (when I am writing the text) I do not know whether the implication "$\widetilde{AP}_{s,\infty} \implies AP_{s,\infty}$" is true, for Banach spaces. Of course, no questions about the cases where $0 < s \le 2/3$ (but the reason is only that every Banach space has the $\widetilde{AP}_{2/3,\infty}$ and the $AP_{2/3,\infty}$).

Let $0 < r < 1$ and $0 < w \le \infty$. or $r = 1$ and $0 < w \le 1$. For Banach spaces X, Y denote by $Y^*\widehat{\otimes}_{(r,w)}X$ the subset of $Y^*\widehat{\otimes}X$ consisting of tensors z such that

$$z = \sum_{k=1}^{\infty} \lambda_k\, y'_k \otimes x_k, \text{ where } (y'_k) \text{ and } (x_k) \text{ are bounded and } (\lambda_k) \in l_{rw}.$$

As was noted in Remark 3.1, if $s \in (0,1), q \in (0,\infty), 1/s = 1 + 1/q$, then $l_1 \cdot l_{q\infty} = l_{s1}$ (in the sense of the product in Lemma 3.1). In general case, where $0 < q_1, q_2, t_1, t_2 \le \infty$, one has

$$(3) \qquad l_{q_1t_1} \cdot l_{q_2t_2} \text{ provided that: } \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{s} \text{ and } \frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}.$$

We can introduce a new definition of approximation properties, which are connected with Lorentz sequence spaces, namely: Let $0 < r < 1$ and $0 < w \le \infty$. or $r = 1$ and $0 < w \le 1$. A Banach space X has the $AP_{(r,w)}$, if for every Banach space Y the natural map $Y^*\widehat{\otimes}_{(r,w)}X \to L(Y,X)$ is one-to-one.

It follows (from Remark 3.1 or from (3)) that $AP_{1;q,\infty} = AP_{(s,1)}$ (for $s \in (0,1)$ and $1/s = 1 + 1/q$) and, more generally, $AP_{t;p,r} = AP_{(s,u)}$ for $1/t + 1/p = 1/s$ and $1/t + 1/r = 1/u$ $(t \in (0,1])$.

Therefore, we have (for $s \in (0,1)$)

$$AP_{s,\infty} \implies AP_{(s,1)} \implies \widetilde{AP}_{s,\infty}.$$

Moreover, taking into account the equality $\widehat{\otimes}_{1;q,\infty} = \widehat{\otimes}_{(s,1)}$ and applying the arguments from the proof of the implication "$\widetilde{AP}_s \implies AP_s$" of Theorem 2.1, we easily get

Theorem 3.3. $AP_{(s,1)} = \widetilde{AP}_{s,\infty}$.

Proof. As was mentioned above, $AP_{(s,1)} \implies \widetilde{AP}_{s,\infty}$. Let X has the $\widetilde{AP}_{s,\infty}$, i.e. for every sequence $(x_n) \in l^0_{q,\infty}$ (where $1/s = 1 + 1/q$) and every $\varepsilon > 0$ there exists a finite rank operator $R \in X^* \otimes X$ such that $\sup_n ||Rx_n - x_n|| < \varepsilon$. Since $AP_{(s,1)} = AP_{1;q,\infty}$, it is enough to show that if Y is a Banach space, $z \in Y^*\widehat{\otimes}_{1;q,\infty}X$ and $z \ne 0$, then the corresponding operator $\widetilde{z} : Y \to X$ is not zero too.

Let $z = \sum_{k=1}^{\infty} a_k b_k\, y'_k \otimes x_k$ be a representation of z with $(x_k), (y'_k)$ bounded, $(a_k) \in l_1$, $(b_k) \in l^0_{q\infty}$ and $b_k \searrow 0$. Then $(\widetilde{x}_k := b_k x_k) \in l^0_{q\infty}$ and, for an $\varepsilon > 0$ small enough (to be he chosen), we can find an operator $R \in X^* \otimes X$ with the property that $\sup_n ||R\widetilde{x}_n - \widetilde{x}_n|| \le \varepsilon$. Since $z \neq 0$, we can find an operator $V \in L(Y^*, X^*)$ such that $\sum_{k=1}^\infty a_k \langle Vy_k', \widetilde{x}_k\rangle = 1$. Now, when V is chosen, we have

$$1 = \sum_{k=1}^\infty a_k \langle Vy_k', \widetilde{x}_k - R\widetilde{x}_k\rangle + \sum_{k=1}^\infty a_k \langle Vy_k', R\widetilde{x}_k\rangle$$

$$\le \varepsilon\, ||(a_k)||_{l_1}\, ||V|| \cdot const + |\sum_{k=1}^\infty a_k b_k \langle R^*Vy_k', x_k\rangle|,$$

and, if ε is small enogh, we get for the finite rank operator $R^*V : Y^* \to X^*$ that

$$|\,\mathrm{trace}\; z^t \circ (R^*V)| = |\,\mathrm{trace}\,(R^*V) \circ z^t| = |\sum_{k=1}^\infty a_k b_k \langle R^*Vy_k', x_k\rangle| > 0.$$

The last sum is the nuclear trace of the tensor element $\sum_{k=1}^\infty a_k b_k\, R^*Vy_k' \otimes x_k$, which is a composition $R \circ z_0$ of the finite rank operator R and the tensor element $\sum_{k=1}^\infty a_k b_k\, Vy_k' \otimes x_k$, that belongs to the tensor product $X^*\widehat{\otimes}_{1;q,\infty}X$. It follows that both z_0 and z generate the non-zero operators $\widetilde{z}_0$ and $\widetilde{z}$.

Remark 3.4. Because of the equality $\widehat{\otimes}_{1;q,\infty} = \widehat{\otimes}_{(s,1)}$, it follows from the proof of Theorem 3.3 that X has the $AP_{(s,1)}$ iff the canonical mapping $X^*\widehat{\otimes}_{(s,1)}X \to L(X)$ is one-to-one (just like in the case of the classical Grothendieck approximation property).

Remark 3.5. Of course, it follows from Theorem 3.3 that every Banach space has the $AP_{(2/3,1)}$, but it is trivial because of the implication

$$AP^0_{(2/3,\infty)} \equiv AP_{2/3,\infty} \implies AP_{(2/3,w)} \text{ for any } w < \infty$$

(and, again, since every X has the $AP_{2/3,\infty}$!).

Our question in Remark 3.3 can be reformulated now as:
(*) Is it true that the $AP_{(s,1)}$ implies the $AP_{s,\infty}$?

IV. Let us consider an application of the previous considerations. Now we know, in particular, that every Banach space has the $AP_{(2/3,1)}$. On the other hand, the corresponding operator ideal $N_{(2/3,1)}$ (related to the Lorentz space $l_{2/3\,1}$) has the eigenvalue type l_1 (see, e.g., [4, p. 243]). Since the continuous trace is unique on $\widehat{\otimes}_{(2/3,1)}$ and $\widehat{\otimes}_{(2/3,1)} = N_{(2/3,1)}$, it follows from White's results [17] that for each Banach space X and for every operator $T \in N_{(2/3,1)}(X,X)$ the (nuclear) trace of T is well defined and equals the sum of all eigenvalues of T :

$$\mathrm{trace}\; T = \sum_{k=1}^\infty \mu_k(T) \text{ (eigenvalues)} \,\forall\, X,\ \forall\, T \in N_{(2/3,1)}(X)$$

(on the right is the so-called "spectral sum" of T). More precisely, the last statement follows from Theorem 4.1 below.

Let us explain in more details how we apply a White's result. For this we formulate and prove a theorem which is almost immediate consequence of the White's theorem.

Theorem 4.1. Let A be a quasi-Banach operator ideal, X be a Banach space, for which the set of all finite rank operators is dense in the space $A(X)$. Suppose that the natural functional trace is bounded on the subspace of all finite rank operators of $A(X)$ (and, therefore, can be extended to a continuous functional "trace_A" on the whole space $A(X)$). If the quasi-Banach operator ideal A is of eigenvalue type l_1, then the spectral trace (= "spectral sum") is continuous on the space $A(X)$ and for any operator $T \in A(X)$ we have

$$\text{trace}_A(T) = \sum_{n=1}^{\infty} \mu_n(T).$$

where $(\mu_n(T))_{n=1}^{\infty}$ is the sequence of all eigenvalues of T, counting by multiplicities.

Proof of Theorem 4.1. Let $T \in A(X)$. By the assumption, the sequence $\{\mu_n(T)\}_{n=1}^{\infty}$ of all eigenvalues of T, counting by multiplicities, is in l_1.

Since the quasi-normed ideal A is of spectral (= eigenvalue) type l_1, we can apply the main result from the paper [17] of M.C. White, which asserts:

$({}^*{}_*{}^*)$ *If J is a quasi-Banach operator ideal with eigenvalue type l_1, then the spectral sum is a trace on that ideal J.*

Recall (see [12], 6.5.1.1, or Definition 2.1 in [17]) that a *trace* on an operator ideal J is a class of complex-valued functions, all of which one writes as τ, one for each component $J(E, E)$, where E is a Banach space, so that

(i) $\tau(e' \otimes e) = \langle e', e\rangle$ for all $e' \in E^*, e \in E$;

(ii) $\tau(AU) = \tau(UA)$ for all Banach spaces F and operators $U \in J(E, F) and A \in L(F, E)$;

(iii) $\tau(S + U) = \tau(S) + \tau(U)$ for all $S, U \in J(E, E)$;

(iv) $\tau(\lambda U) = \lambda\tau(U)$ for all $\lambda \in \mathbb{C}$ and $U \in J(E, E)$.

Our operator T belongs to the space $A(X, X) = A(X)$ and A is of eigenvalue type l_1. Thus, the assertion $({}^*{}_*{}^*)$ implies that the spectral sum λ, defined by $\lambda(U) := \sum_{n=1}^{\infty} \lambda_n(U)$ for $U \in A(E, E)$, is a trace on A.

By principle of uniform boundedness (see [11], 3.4.6 (page 152), or [9]), there exists a constant $C > 0$ with the property that

$$|\lambda(U)| \le ||\{\lambda_n(U)\}||_{l_1} \le C\, a(U)$$

for all Banach spaces E and operators $U \in A(E, E)$.

Now, remembering that all operators in $A(X)$ can be approximated by finite rank operators and taking in account the conditions (iii)–(iv) for $\tau = \lambda$, we obtain that the A-trace, i.e. $\text{trace}_A T$, of our operator T coincides with $\lambda(T)$ (recall that the continuous trace is uniquely defined in such a situation, that is on the space $A(X)$; cf. [12], 6.5.1.2).

Since $\widehat{\otimes}_{1;2,\infty} = \widehat{\otimes}_{(2/3,1)}$, (see Theorem 3.3), we can reformulate the result, which we formulated in the very beginning of this section, as

Corrolary 4.1. For each Banach space X and for every operator $T \in N_{1;2,\infty}(X, X)$ the (nuclear) trace of T is well defined and equals the sum of all eigenvalues of T :

$$\text{trace } T = \sum_{k=1}^{\infty} \mu_k(T) \text{ (eigenvalues)} \, \forall\, X,\ \forall\, T \in N_{(1;2,\infty)}(X).$$

Remark 4.1: Recall that A. Grothendieck [3] has obtained the assertion of the last fact for the case of 2/3-nuclear operators, i.e. for the ideal $N_{2/3} = N_{(2/3,\,2/3)}$ (note that $l_{2/3} \subset l_{2/3\,1}$).

V. The discussion on Section I shows that, for $p \in [1, \infty]$, any subspace of any quotient (= any quotient of any subspace) of an L_p-space possesses the $\widetilde{AP}_s$ (even the $\widetilde{AP}_{s,\infty}$) with $1/s = 1 + |1/2 - 1/p|$. We apply now these facts together with the White theorem for proving some more theorems concerning the distributions of eigenvalues of the nuclear operators. Below we will use Theorem 2.1 and, therefore, the fact that any subspace of any quotient of an L_p-space possesses the AP_s (where p, s as above). Thus, for such Banach spaces X, we can identify the tensor product $X^* \widehat{\otimes}_s X$ with its canonical image in the space $L(X) = L(X, X)$, that is with the space $N_s(X)$ of all s-nuclear operators in X, equipped with the quasi-norm induced from $X^* \widehat{\otimes}_s X$.

We are going to give below the relatively simple proofs of some recent results from the papers [15] and [16]. Let us begin.

Theorem 5.1. Let X be a subspace of an L_p-space, $1 \le p \le \infty$. If $T \in N_s(X, X)$, where $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of T is well defined,
2. $\sum_{n=1}^{\infty} |\mu_n(T)| < \infty$, where $\{\mu_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities)

and

$$\text{trace } T = \sum_{n=1}^{\infty} \mu_n(T).$$

Proof. Let X be a subspace of an L_p-space $L_p(\mu)$ and $T \in N_s(X, X)$ with an s-nuclear representation

$$T = \sum_{k=1}^{\infty} \lambda_k x'_k \otimes x_k,$$

where $||x'_k||, ||x_k|| = 1$ and $\lambda_k \ge 0$, $\sum_{k=1}^{\infty} \lambda_k^s < \infty$. By Hahn-Banach, we can find the functionals $\widetilde{x}'_k \in L^*_p(/mu)$ $(k = 1, 2, \dots)$ with the same norms as the corresponding functionals x'_k and so that $\widetilde{x}'_k|_X = x'_k$ for every k. Denote by $\widetilde{T}$ the operator

$$\widetilde{T} : L_p(\mu) \to X,\ \widetilde{T} := \sum_{k=1}^{\infty} \lambda_k \widetilde{x}'_k \otimes x_k,$$

and by $j : X \to L_p(mu)$ the natural injection. Since the space X has the property AP_s, we have $N_s(L_p(\mu), X) = L^*_p(\mu) \widehat{\otimes}_s X$ and, therefore, the nuclear traces of the operators $j\widetilde{T}$ and $\widetilde{T}j$ are well defined. We have a diagram

$$X \xrightarrow{j} L_p(\mu) \xrightarrow{\widetilde{T}} X \xrightarrow{j} L_p(\mu),$$

in which $\widetilde{T}j = T \in N_s(X)$. Hence, the complete systems of eigenvalues of the operators $T = \widetilde{T}j$ and $j\widetilde{T} \in N_s(L_p(\mu))$ coincide . Applying Theorem 2.b.13 from [5] (see also [15]), we obtain that the sequence $(\mu_k(j\widetilde{T}))$ is summable. Therefore, we have $\mu_k(T) \in l_1$ and we can apply Theorem 4.1. But we apply the theorem firstly for the simplest case (later on we will continue the proof of our theorem 5.1).

The first assertion of the next theorem is due to A. Grothendieck [3], the second one was proved by H,. König in [6]. Surprisingly, but we could not find anywhere the main statement of the theorem about coincidence of the nuclear and spectral traces, neither in the monographs, nor in the mathematical journals. So we have no reference for this statement and have to formulate and to prove the next theorem here. Let us remark that, in any case, this theorem was proved (as a partial case of the proved there our Theorem 5.1) in [15].

Theorem 5.1'. Let L be an L_p-space, $1 \leq p \leq \infty$. If $T \in N_s(L, L)$, where $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of T is well defined,
2. $\sum_{n=1}^{\infty} |\mu_n(T)| < \infty$, where $\{\mu_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities)

and

$$\text{trace } T = \sum_{n=1}^{\infty} \mu_n(T).$$

Proof. As was said above, the assertions 1 and 2 are well known. To prove the last equality, consider the Banach operator ideal $\mathcal{L}_p$ of all operators which can be factored through L_p-spaces. Then the product $\mathcal{L}_p \circ N_s$ is a quasi-Banach operator ideal of spectral (=eigenvalue) type l_1 (e.g., by the assertion 2, proved earlier by H. König [6]). Now it is enough to apply Theorem 4.1 to finish the proof.

Proof of Theorem 5.1 (continuation). As was said, the complete systems of eigenvalues of the operators $T = \widetilde{T}j$ and $j\widetilde{T} \in N_s(L_p(\mu))$ coincide. By Theorem 5.1',

$$\text{trace } j\widetilde{T} = \sum_{k=1}^{\infty} \lambda_k \langle \widetilde{x}'_k, jx_k \rangle = \sum_{n=1}^{\infty} \mu_n(j\widetilde{T}),$$

the last sum is equal to

$$\sum_{n=1}^{\infty} \mu_n(T)$$

and the sum in the middle is

$$\sum_{k=1}^{\infty} \lambda_k \langle \widetilde{x}'_k, jx_k \rangle = \sum_{k=1}^{\infty} \lambda_k \langle x'_k, x_k \rangle = \text{trace } T.$$

The (nuclear) trace of the operator T is well defined, because the space X has the AP_s. Therefore,

$$\text{trace } T = \sum_{n=1}^{\infty} \mu_n(T),$$

and we are done.

If Y is a quotient of an L_p-space, then, for a compact operator $U \in L(E, E)$, the adjoint U^* is also a compact operator and these two operators have the same eigenvalues $\mu \neq 0$ with the same multiplicities (see, e.g., [11], Theorem 3.2.26, or [2], Exercise VII.5.35). Also, any quotient of an L_p-space has the AP_s (where p, s are as above). So, it follows immediately from the just proved Theorem 5.1

Corrolary 5.1. Let Y be a quotient of an L_p-space, $1 \leq p \leq \infty$. If $T \in N_s(Y, Y)$, where $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of T is well defined,
2. $\sum_{n=1}^{\infty} |\mu_n(T)| < \infty$, where $\{\mu_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities)

and

$$\text{trace } T = \sum_{n=1}^{\infty} \mu_n(T).$$

We used above some facts from the section I. After Theorem 5.1 and its consequence are proved, we are ready to present a simple prove of the corresponding result on the subspaces of quotients of the L_p-spaces (recall that, again, all such spaces have the AP_s with s and p satisfying the same conditions).

Theorem 5.2. Let W be a quotient of a subspace (= a subspace of a quotient) of an L_p-space, $1 \leq p \leq \infty$. If $T \in N_s(W, W)$, where $1/s = 1 + |1/2 - 1/p|$, then

1. the (nuclear) trace of T is well defined,
2. $\sum_{n=1}^{\infty} |\mu_n(T)| < \infty$, where $\{\mu_n(T)\}$ is the system of all eigenvalues of the operator T (written in according to their algebraic multiplicities)

and

$$\text{trace } T = \sum_{n=1}^{\infty} \mu_n(T).$$

Proof. Let $L_p(\mu)$ be an L_p-space. Take Banach subspaces $X_0 \subset X \subset L_p(\mu)$ and consider the quotient X/X_0. If $T \in N_s(X/X_0, X/X_0)$ $(=(X/X_0)^* \widehat{\otimes}_s X/X_0)$, then T admits a factorization of the type

$$X/X_0 \xrightarrow{A} c_0 \xrightarrow{D} l_1 \xrightarrow{B} X/X_0,$$

where A, B are continuous and D is a diagonal operator with a diagonal from l_s.

Denoting by $\varphi : X \to X/X_0$ the factor map from X onto X/X_0 and taking a lifting $\Phi : l_1 \to X$ for B with $B = \varphi\Phi$, we obtain that the maps $\varphi\Phi DA : X/X_0 \to X/X_0$ and $\Phi DA\varphi : X \to X$ have the same eigenvalues $\mu \neq 0$ with the same multiplicities:

$$X \xrightarrow{\varphi} X/X_0 \xrightarrow{A} c_0 \xrightarrow{D} l_1 \xrightarrow{\Phi} X \xrightarrow{\varphi} X/X_0,$$

The spaces X and X/X_0 have the AP_s. Therefore, we have (cf. the proof of Theorem 5.1)

$$\text{trace } \varphi\Phi DA = \text{trace } \Phi DA\varphi.$$

Since X is a subspace of $L_p(\mu)$, we have, by Theorem 5.1,

$$\text{trace } \Phi DA\varphi = \sum_{n=1}^{\infty} \mu_n(\Phi DA\varphi).$$

Therefore,

$$\text{trace } T = \text{trace } BDA = \text{trace } \varphi\Phi DA = \sum_{n=1}^{\infty} \mu_n(\Phi DA\varphi)$$

$$= \sum_{n=1}^{\infty} \mu_n(\varphi\Phi DA) = \sum_{n=1}^{\infty} \mu_n(BDA) = \text{trace } T.$$

VI. As is well known, in the classical case of the Grothendieck approximation property AP if X^* has the AP, then the space X also has this property. We will show now that the same is true for all approximation properties which are under consideration in this paper.

Denote by $\widehat{\otimes}_\alpha$ any of the tensor product $\widehat{\otimes}_s$, $\widehat{\otimes}_{s,\infty}$, $\widehat{\otimes}_{t;p,r}$, $\widehat{\otimes}_{(r,w)}$ with the parameters (see above), for which all those tensor products are the linear subspaces of the projective tensor product $\widehat{\otimes}$. Also, let us say that a Bansch space X has the AP_α, if it is possesses the corresponding approximation property (i.e., AP_s, $AP_{s,\infty}$ etc.).

We need the following auxiliary result which may be of its own interest (compare with Remark 3.4).

Proposition 6.1 A Banach space X has the AP_α iff the canonical map $X^*\widehat{\otimes}_\alpha X \to L(X)$ is one-to-one.

Proof. Suppose that the canonical map $X^*\widehat{\otimes}_\alpha X \to L(X)$ is one-to-one, but there exista a Banach space Y such that the natural map $Y^*\widehat{\otimes}_\alpha X \to L(X)$ is not injective. Let $z \in Y^*\widehat{\otimes}_\alpha X \to L(X)$ be such that $z \neq 0$ and the assosoated operator $\widetilde{z}$ is a 0-operator. Then we can find an operator V from $L(Y^*, X^*)$ (the dual space to the projective tensor product $Y^*\widehat{\otimes}X$) so that trace $V \circ z^t = 1$, where, as usual, z^t is the transposed tensor element, $z * t \in X\widehat{\otimes}Y^*$. Since $V \circ z^t \in X\widehat{\otimes}X^*$ and trace $V \circ z^t = 1$, the tensor element $(V \circ z^t)^t$ (which, evidently, belongs to $X^*\widehat{\otimes}_\alpha X$) is not zero. On the other hand, the operator induced by this element must be a 0-operator. Contradiction.

Proposition 6.2. With the above understanding, if the dual space X^* has the AP_α, then X has the AP_α too.

Proof. We use Proposition 6.1. As is known [5], the projective tensor product $Y^*\widehat{\otimes}Y$ is a Banach subspace of the tensor product $Y^*\widehat{\otimes}Y^{**}$. The tensor product $Y^*\widehat{\otimes}_\alpha Y$ is a linear subspace of $Y^*\widehat{\otimes}Y$, as well as $Y^*\widehat{\otimes}_\alpha Y^{**}$ is a linear subspace of $Y^*\widehat{\otimes}Y^{**}$. Therefore, the natural map $Y^*\widehat{\otimes}_\alpha Y \to Y^*\widehat{\otimes}_\alpha Y^{**}$ is one-to-one. Now if Y^* has the AP_α, then the canonical map $Y^{**}\widehat{\otimes}_\alpha Y^* \to L(Y^*, Y^*)$ is one-to-one. Since we can identify the tensor product $Y^{**}\widehat{\otimes}_\alpha Y^*$ with the tensor product $Y^*\widehat{\otimes}_\alpha Y^{**}$ (because of the "symmetries" in the definitions of the corresponding tensor products), it follows that the natural map $Y^*\widehat{\otimes}_\alpha Y \to L(Y,Y)$ is one-to-one. Thus, if Y^* has the AP_α, then Y has the AP_α too.

Remark 2: The inverse statement is not true. For example, if $s \in (2/3, 1]$, then there exists a Banach space, possessing the Grothendieck approximation property, whose dual does not have the AP_s (it is well known for the case where $s = 1$). Moreover, if $s \in (2/3, 1]$, then we can find a Banach space W such that W has a Schauder basis and W^* does not have the AP_s. Indeed, let E be a separable reflexive Banach space without the AP_s (see [7] or [8]). Let Z be a separable space such that Z^{**} has a basis and there exists a linear homomorphism φ from Z^{**} onto E^* so that the subspace $\varphi^*(E)$ is complemented in Z^{***} and, moreover, $Z^{***} \cong \varphi^*(E) \oplus Z^*$ (see [7, Proof of Corollary 1]). Put $W := Z^{**}$. This (second dual) space W has a Schauder basis and its dual W^* does not have the AP_s.

VII. Let us consider some more notions of the approximation properties assosoated with some other tensor products. For Banach spaces X and Y and $r \in (0,1], p \in [1,2]$, define a quasi-norm $||\cdot||_{N_{[r,p]}}$ on the tensor product $X^* \otimes Y$ by

$$\|u\|_{N_{[r,p]}} := \inf \left\{ \|(x_i')_{i=1}^n\|_{\ell_r(X^*)} \cdot \|(y_i)_{i=1}^n\|_{\ell^w_{p'}(Y)} :\ u = \sum_{i=1}^n x_i' \otimes y_i \right\}$$

Here we denote, as usual, by $l_r(X^*)$ and $l^w_q(Y)$ the spaces of r-absolutely summable and weakly q-summable sequences, respectively.

Denote by через $X^*\widehat{\otimes}_{[r,p]}Y$ the completion of the space $\left(X^* \otimes Y,\ \|\cdot\|_{N_{[r,p]}}\right)$. We have a natural continuous injection

$$j_{[r,p]} : X^*\widehat{\otimes}_{[r,p]}Y \to X^*\widehat{\otimes}Y$$

with $||j_{[r,p]}|| \le 1$.

Every element $u \in X^*\widehat{\otimes}_{[r,p]}Y$ has a representation of the type $u = \sum_{i=1}^\infty x_i' \otimes y_i$, where $(x_i')_{i=1}^\infty \in \ell_r(X^*)$ and $(y_i)_{i=1}^\infty \in \ell^w_{p'}(Y)$. Consider the natural mappings

$$X^*\widehat{\otimes}_{[r,p]}Y \overset{\widetilde{j}_{[r,p]}}{\to} X^*\widehat{\otimes}Y\widetilde{j} \to L(X,Y).$$

The image of the tensor product $X^*\widehat{\otimes}_{[r,p]}Y$ under the composition $\widetilde{j}_{[r,p]} := \widetilde{j} \circ \widetilde{j}_{[r,p]}$ is denoted by $N_{[r,p]}(X,Y)$. This is a quasi-Banach space of the (r,p)-nuclear operators (the quasi-norm is induced from the tensor produce $X^*\widehat{\otimes}_{[r,p]}Y$). It is not difficult to see that every operator $T \in N_{[r,p]}(X,Y)$ admit a factorization of the kind

$$X \overset{A}{\to} c_0 \overset{D_r}{\to} l_1 \overset{i}{\to} l_p \overset{B}{\to} Y,$$

where A, B are compact, i is the injection, D_r is a diagonal operator with a diagonal from l_r. Also, every operator which can be factored in such a way is in $T \in N_{[r,p]}(X,Y)$.

By the analogous way, we define the tensor product $X^*\widehat{\otimes}^{[r,p]}Y$ and the quasi-normed operator ideals $N^{[r,p]}(X,Y)$. Namely, $X^*\widehat{\otimes}^{[r,p]}Y$ is a linear subapace of the projective tensor product $X^*\widehat{\otimes}Y$, consisting of tensor elements z which admit a representation

$$u = \sum_{i=1}^{\infty} x_i' \otimes y_i,$$

where $(x_i')_{i=1}^{\infty} \in \ell_{p'}^w(X^*)$ and $(x_i)_{i=1}^{\infty} \in \ell_r(Y)$. Its canonical image in $L(X,Y)$ is the quasi-normed space $N^{[r,p]}(X,Y)$. It is not difficult to see that every operator $T \in N^{[r,p]}(X,Y)$ admit a factorization of the kind

$$X \xrightarrow{A} l_{p'} \xrightarrow{D_r} c_0 \xrightarrow{i} l_1 \xrightarrow{B} Y,$$

where A, B are compact, i is the injection, D_r is a diagonal operator with a diagonal from l_r. Also, every operator which can be factored in such a way is in $T \in N^{[r,p]}(X,Y)$.

It is clear that $T^* \in N_{[r,p]}(Y^*,X^*)$ implies $T \in N^{[r,p]}(X,Y)$ and $T^* \in N^{[r,p]}(Y^*,X^*)$ implies $T \in N_{[r,p]}(X,Y)$.

Now we can define the notions of the corresponding approximation properties by the usual way. We say that he space X has the $AP_{[r,p]}$ (respectively, the $AP^{[r,p]}$) if for every Banach space Y the natural mapping $Y^*\widehat{\otimes}_{[r,p]}X \to L(Y,X)$ (respectively, $Y^*\widehat{\otimes}^{[r,p]}X \to L(Y,X)$) is one-to-one. It can be seen that a Banach space X has the $AP_{[r,p]}$ (or $AP^{[r,p]}$) iff the canonical map $X^*\widehat{\otimes}_{[r,p]}X \to L(X)$ (or $X^*\widehat{\otimes}^{[r,p]}X \to L(X)$) is one-to-one (the proof is essentially the same as the proof of Theorem 6.1). Also, if X^* has the $AP_{[r,p]}$ (or $AP^{[r,p]}$) then X has the $AP^{[r,p]}$ (or $AP_{[r,p]}$) (the proof is the as in Theorem 6.2).

Theorem 7.1. Let $1/r - 1/p = 1/2$. Every Banach space has the properties $AP_{[r,p]}$ and $AP^{[r,p]}$.

Proof. Suppose that $X \notin AP_{[r,p]}$ where $1/r - 1/p = 1/2$. Let $z \in X^*\widehat{\otimes}_{[r,p]}X$ be a n element such that trace $z = 1, \tilde{z} = 0$. Since $z = \sum x_k' \otimes x_k$, where $(x_k') \in l_r(X^*)$ and (x_k) is weakly p'-summable, the operator $\tilde{z}$ can be factored as

$$\tilde{z}:\ X \xrightarrow{A} l_\infty \xrightarrow{\Delta} l_1 \xrightarrow{j} l_p \xrightarrow{V} X,$$

where all the operators are continuous, , j is an injection, Δ is a diagonal operator with a diagonal from l_r. Since $\tilde{z} = 0$, we have $V|_{j\Delta A(X)} = 0$. Consider $S := j\Delta AV : l_p \to l_p$. Evidently, $S^2 = 0$ и trace $S =$ trace $z = 1$. Since $S \in N_r(l_p,l_p)$, its nuclear trace equals the sum of all its eigenvalues (see Theorem 5.1' above). This contradicts the fact that $S^2 = 0$.

We are ready to apply the above results to the investigation of eigenvalues problems for $N_{[r,p]}$- and $N^{[r,p]}$-operators. The first theorem below was proved in [16] by using Fredholm Theory. The same proof can be applied for the second theorem.

Theorem 7.2. Let $1/r - 1/p = 1/2$. For every Banach space X and every operator $T \in N_{r,p}(X)$, trace(T) is well defined and if $(\mu_i)_{i=1}^{\infty}$ is a system of all eigenvalues of T, then $(\mu_i)_{i=1}^\infty \in l_1$ and

$$\operatorname{trace}(T) = \sum_{i=1}^\infty \mu_i.$$

Theorem 7.3. Let $1/r - 1/p = 1/2$. For every Banach space X and every operator $T \in N^{r,p}(X)$, trace (T) is well defined and if $(\mu_i)_{i=1}^\infty$ is a system of all eigenvalues of T, then $(\mu_i)_{i=1}^\infty \in l_1$ and

$$\operatorname{trace}(T) = \sum_{i=1}^\infty \mu_i.$$

Both theorems can be proved by the analogues methods and the proofs are almost the same as the proof of Theorem 5.2 (by using Theorem 7.1). So we omit it here.

VIII. The next examples are taken from [16], where one can find the corresponding proofs. They show that all the above positive results concerning approximation properties and trace-formulas are sharp.

Example 8.1. Let $r \in (2/3, 1], p \in (1, 2], 1/r - 1/p = 1/2$. There exist Banach spaces E and V, $z_0 \in E^*\widehat{\otimes}V, S \in L(V, E)$ so that for every $p_0 \in [1, p)$

1) $z_0 \in E^*\widehat{\otimes}_{r,1}V$;
2) V has a basis;
3) V is the space of type p_0 and of cotype 2;
4) $S \circ z_0 \in E^*\widehat{\otimes}_{r,p_0}E$;
5) trace $S \circ z_0 = 1$;
6) the corresponding operator $\widetilde{S \circ z_0}$ is a 0-operator and, therefore, has no nonzero eigenvalues.

Example 8.2. Let $r \in (2/3, 1), p \in (1, 2], 1/r - 1/p = 1/2$. There exist Banach spaces E and V, $z_0 \in E^*\widehat{\otimes}V, S \in L(V, E)$ so that for every $\epsilon > 0$

1) $z_0 \in E^*\widehat{\otimes}_{r+\epsilon,1}V$;
2) V has a basis;
3) $S \circ z_0 \in E^*\widehat{\otimes}_{r+\epsilon,p}E$;
4) trace $S \circ z_0 = 1$;
5) the corresponding operator $\widetilde{S \circ z_0}$ is a 0-operator and therefore, has no nonzero eigenvalues.

Example 8.3. Let $r \in (2/3, 1]$, $p \in (1, 2], 1/r - 1/p = 1/2$. There exist two separable Banach spaces X and Z so that

(i) Z^{**} has a basis;
(ii) $\exists\, V \in X^*\widehat{\otimes}Z^{**} : V = \sum_{k=1}^\infty x'_k \otimes z''_k$; (x'_k) weakly p'_0-summable for each $p_0 \in [1, p)$; $(z''_k) \in l_r(Z^{**})$;
(iii) $V(X) \subset Z$; the operator V is not nuclear as a map from X into Z.

Moreover, there exista an operator $U : Z^{**} \to Z$ such that

(α) $\pi_Z U \in N^{r,p_0}(Z^{**}, Z^{**}) = Z^{***}\widehat{\otimes}^{r,p_0}Z^{**}$, $\forall\, p_) \in [1, p)$;
(β) U is not nuclear as a map from Z^{**} into Z;

(γ) trace $\pi_Z U = 1$;
(δ) $\pi_Z U : Z^{**} \to Z^{**}$ has no nonzero eigenvalues.

References

[1] J. Bergh, J. Lofstrom: *Interpolation spaces*, Berlin, Heidelberg, New York: Springer-Verlag (1976).
[2] N. Dunford, J. Schwartz: *Linear Operators, Part I*, Interscience, New York (1958).
[3] A. Grothendieck: *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc., **16**(1955).
[4] A. Hinrichs, A. Pietsch, p-nuclear operators in the sense of Grothendieck, Math. Nachr., Volume 283, No. 2 (2010), 232–261.
[5] H. Konig: *Eigenvalue Distribution of Compact Operators*, Birkhäuser (1986).
[6] H. Konig: *On the eigenvalue spectrum of certain operator ideals*, Coll. Math. 44 (1981), 15–28.
[7] J. Lindenstrauss, On James' paper "Separable Conjugate Spaces", Israel J. Math., Volume 9 (1971), 279–284.
[8] J. Lindenstrauss, L. Tzafriri: *Classical Banach spaces, vol.1: Sequence spaces*, Berlin-Heidelberg-New York (1977).
[9] A. Pietsch: *Eigenwertverteilungen von Operatoren in Banachrdumen*, Hausdorff-Festband: Theory of sets and topology, Berlin: Akademie-Verlag (1972), 391-402.
[10] A. Pietsch: *Operator Ideals*, North Holland (1980).
[11] A. Pietsch: *Eigenvalues and s-numbers*, Cambridge Univ. Press (1987).
[12] A. Pietsch: *History of Banach Spaces and Linear Operators*, Birkhäuser (2007).
[13] O.I. Reinov: *Approximation properties AP_s and p-nuclear operators (the case $0 < s \leq 1$)*, Journal of Mathematical Sciences **115**, No. 3 (2003), 2243-2250.
[14] O.I. Reinov: *Some More Remarks on Grothendieck-Lidskii Trace Formulas*, Journal of Prime Research in Mathematics **8** (2012), 148-154.
[15] Oleg Reinov, Qaisar Latif: *Grothendieck-Lidskii theorem for subspaces of L_p-spaces*, Math. Nachr. **286**, No. 2-3 (2013), 279–282.
[16] Oleg Reinov, Qaisar Latif: *Distribution of eigenvalues of nuclear operators and Grothendieck–Lidskii type formulas*, Journal of Mathematical Sciences **193**, No. 2 (2013), 312–329.
[17] M.C. White: *Analytic multivalued functions and spectral trace*, Math. Ann. 304 (1996), 665-683.

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