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Geometrical Characterization of RN-operators between Locally Convex Vector Spaces

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Abstract: For locally convex vector spaces (l.c.v.s.) E and F and for linear and continuous operator $T : E \to F$ and for an absolutely convex neighborhood V of zero in F, a bounded subset B of E is said to be T-*V-dentable* (respectively, T-*V-s-dentable*, respectively, T-*V-f-dentable*) if for any $\epsilon > 0$ there exists an $x \in B$ so that

$$x \notin \overline{co}\,(B \setminus T^{-1}(T(x) + \epsilon V))$$

(respectively, so that $x \notin s\text{-}co\,(B \setminus T^{-1}(T(x) + \epsilon V))$, respectively, so that $x \notin co\,(B \setminus T^{-1}(T(x) + \epsilon V))$). Moreover, B is called T-*dentable* (respectively, T-*s-dentable*, T-*f-dentable*) if it is T-*V-dentable* (respectively, T-*V-s-dentable*, T-*V-f-dentable*) for every absolutely convex neighborhood V of zero in F.RN-operators between locally convex vector spaces have been introduced in [5]. We present a theorem which says that, for a large class of l.c.v.s. E, F, if $T : E \to F$ is a linear continuous map, then the following are equivalent:
1) $T \in RN(E, F)$.
2) Each bounded set in E is T-dentable.
3) Each bounded set in E is T-s-dentable.
4) Each bounded set in E is T-f-dentable.
Therefore, we have a generalization of Theorem 1 in [8], which gave a geometric characterization of RN-operators between Banach spaces.

Key–Words: dentability, dentable set, locally convex space, Radon-Nikodym operator

1 Introduction

We use standard notations and terminology from the theory of operators in Banach spaces and measure theory (see, e.g., [1],[3]). Our main references on the theory of the locally convex vector spaces is [12]. Banach spaces are usually denoted by the letters $X, Y, \ldots$, and the usual notations for the l.c.v.s. (locally convex vector spaces) are $E, F, \ldots$. All the spaces we consider in the paper are over the field of the real numbers.

Some of the main notions and notations we will use are ones of the convex, s-convex and closed convex hulls of sets in Banach or l.c.v. spaces. Namely,

$co(A)$ (respectively, $\overline{co}(A)$, respectively, $s\text{-}co(A)$) denotes the convex hull (respectively, the closure of the convex hull, the s-closure of the convex hull) of the set A. The s-closure is defined as $s\text{-}co(A) := \{\sum\limits_{i\geq 1} a_i x_i$, where $x_i \in A$, $a_i \geq 0$ and $\sum\limits_{i\geq 1} a_i = 1\}$.

Let us recall some more notions:

Definition 1 *Let X be a Banach space and (Ω, Σ) a measureable space, consider $m : \Sigma \to X$. m is called a vector measure if for every sequence $\{A_i\}_1^\infty$ of pairwise disjoint sets from Σ one has $m(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty m(A_i)$.*

Definition 2 *Let $m : \Sigma \to X$ be a vector measure. Variation of a vector measure is a non-negative, extended real valued function with value on set $A \in \Sigma$ is*

$$|m|(A) = \sup_{\prod} \sum_{A_i \in \prod} \|m(A_i)\|$$

where $\prod$ denotes all finite partitions of A with pairwise disjoint sets in Σ; m is called vector measure of bounded variation if $|m|(\Omega)$ is finite.

Let (Ω, Σ, μ) be a measure space.

Definition 3 *A function $f : \Omega \to X$ is μ-measureable if there exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ with $\lim_n \|f_n - f\| = 0$ μ-a.e.* (see [1],page 41).

Definition 4 *A μ-measureable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions f_n such that $\lim_n \int_\Omega \|f_n - f\| d\mu = 0$* (see [1],page 44)*; in this case we set $\int_A f = \lim_n \int_A f_n$ for every $A \in \Sigma$.*

The following notions have been introduced in [9] and, independently, in [6].

Definition 5 *A linear continuous operator $T : X \to Y$ is said to be RN-operator ($T \in RN(X, Y)$), if for every X-valued measure m of bounded variation, for any measure space (Ω, Σ, μ) with $m << \mu$ there exists a function $f : \Omega \to Y$ which is Bochner integrable and such that*

$$T(m)(A) = \int_A f d\mu$$

for every $A \in \Sigma$.

Definition 6 *Let $T : X \to Y$ be a linear continuous operator A bounded subset B of Banach space X is called T-dentable (respectively, T-f-dentable, T-s-dentable), if for every $\epsilon > 0$ there exists $x \in B$ such that*

$$x \notin \overline{co}\,(B \setminus T^{-1}(D_\epsilon(T(x)))).$$

(respectively, so that

$$x \notin co\,(B \setminus T^{-1}(D_\epsilon(T(x))\,)),$$

resp.,

$$x \notin s - co\,(B \setminus T^{-1}(D_\epsilon(T(x))\,))),$$

The geometrical characterization of RN-operators between Banach spaces has been given by following theorem in [8].

Theorem 7 *Let X, Y be Banach spaces and $T : X \to Y$, linear and continuous then the following statements are equivalent:*

1, $T \in RN(X, Y)$
2. Each bounded set in X is T-dentable.
3. Each bounded set in X is T-s-dentable.
4. Each bounded set in X is T-f-dentable.

Our aim is to generalize the above theorem for operators between some l.c.v. spaces. Firstly, we shall define the notion of RN-operators between locally convex vector spaces. For this, we recall the definitions of Banach spaces of type E_B and $\hat{E_V}$. Then we define the notions of T-dentability, T-s-dentability and T-f-dentability for operators in l.c.v.s. (in the next section).

Let E be a l.c.v.s. and $B \subseteq E$ be bounded and absolutely convex. Let $E_B = \bigcup_{n=1}^{\infty} nB$. Define Minkowski function on E_B w.r.t. B :

$$\|x\|_{\rho_B} = \rho_B(x) = \inf\{\lambda \geq 0 : x \in \lambda B\}.$$

It is a seminorm in general but we show that when B is bounded and absolutely convex then it is a norm. Suppose $x \in E_B$ such that $\rho_B(x) = 0$ so $\inf\{\lambda \geq 0 : x \in \lambda B\} = 0$ this implies for any $\epsilon > 0\ \exists\ \lambda \geq 0$ such that $x \in \lambda B$ and $\lambda \leq \epsilon$, take $\epsilon = \frac{1}{n}$, so $x \in \frac{1}{n}B\ \forall\ n \in \mathrm{N}$ this implies $x \in \bigcap_{n=1}^{\infty} \frac{1}{n}B = \{0\}$. So $x = 0$ and (E_B, ρ_B) is a normed space and its completion $\hat{E_B}$ is Banach space in E. Moreover, if B is complete then E_B is Banach space.

Similarly for an absolutely convex open neighborhood of zero $V = V(0)$ we set $E_V = \bigcup_{n=1}^{\infty} nV = E$. Similarly we define Minkowski function on E_V w.r.t V, so that for $x \in E$ the semi norm of x is

$$\|x\|_{\rho_V} = \rho_V(x) = \inf\{\lambda \geq 0 : x \in \lambda V\}$$

.

We identify two elements $x, y \in E$ w.r.t. ρ_V, obtaining a quotient E_V with corresponding elements $\hat{x}, \hat{y}$, by $\hat{x} = \hat{y}$ iff $\rho_V(x - y) = 0$. We get a normed space $\hat{E_V}$ and its completion gives us a Banach space $(\hat{E_V}, \rho_V)$.
We will use the following important theorem (see [10]) in the proof of our main theorem.

Theorem 8 (see [10]) *Let E be a locally convex vector space, $V = V(0)$ be an absolute convex neighborhood of 0, let $B \subseteq E$ be a closed, bounded, convex, sequentially complete and metrizable subset, The following are equivalent:*
(i) B is subset V-dentable.
(ii) B is subset V-f-dentable.

It has been shown in [10] that it follows from the above theorem:

Let E be a locally convex vector space and let $B \subseteq E$ have the following properties:
(i) B is closed, bounded, convex and sequentially complete,
(ii) for every bounded $M \subseteq E$ and for $x \in \overline{M}$ there exist a sequence $x_n \in M$ such that $\lim_n x_n = x$,
(iii) each separable subset of B is metrizable.

Then the following are equivalent:
(i) B is subset V-dentable,
(ii)B is subset V-s-dentable,
(ii)B is subset V-f-dentable.

We will say that *a locally convex vector space E is an SBM-space if*

(i) *every closed bounded convex subset of E is sequentially complete,*

(ii) *for every bounded $M \subseteq E$ and for $x \in \overline{M}$ there exists a sequence $x_n \in M$ such that $\lim_n x_n = x$ and*

(iii) *each separable bounded subset of E is metrizable.*

Therefore, if E is an SBM-space, then every bounded subset of E is V-dentable iff every bounded subset of E is V-s-dentable iff every bounded subset of E is V-f-dentable.

All quasi-complete (BM)-spaces [11], in particular, all Fréchet spaces are SBM-spaces.

2 Main Results

Definition 9 *Let E and F be locally convex vector spaces (l.c.v.s.), let $T : E \to F$ be a linear and continuous operator and let V be an absolutely convex neighborhood of zero in F. A bounded subset B of E is said to be T-V-dentable (respectively, T-V-s-dentable, respectively, T-V-f-dentable) if for any $\epsilon > 0$ there exists an $x \in B$ so that*

$$x \notin \overline{co}\,(B \setminus T^{-1}(T(x) + \epsilon V))$$

(respectively, so that

$$x \notin s - co\,(B \setminus T^{-1}(T(x) + \epsilon V))\,,$$

respectively, so that

$$x \notin co\,(B \setminus T^{-1}(T(x) + \epsilon V))\,).$$

Definition 10 *Let $T : E \to F$ be a linear and continuous operator. A bounded subset B of E is said to be T-dentable (respectively, T-s-dentable, T-f-dentable) if for every absolutely convex neighborhood V of zero in F there exists an $x \in B$ such that*

$$x \notin \overline{co}\,(B \setminus T^{-1}(T(x) + V))$$

(respectively, so that

$$x \notin s - co\,(B \setminus T^{-1}(T(x) + V))$$

and, respectively, so that

$$x \notin co\,(B \setminus T^{-1}(T(x) + V))\,).$$

Remark 11 *From the above two definitions it is clear that $B \subseteq E$ is T-dentable if and only if for every V it is T-V-dentable. The same is true for corresponding properties of s-dentability and f-dentability.*

The following is the definition of an RN-operator between locally convex vector spaces, and this is our main definition.

Definition 12 (see [5]) *Let $T : E \to F$ be linear and continuous (in l.c.v.s.). $T \in RN(E, F)$ (a Radon-Nikodym operator or RN-operator) if for every complete, absolutely convex and bounded set $B \subseteq E$ and for any absolutely convex neighborhood $V \subseteq F$ of zero the natural operator $\Psi_V \circ T \circ \phi_B : \hat{E_B} \to E \to F \to \hat{F_V}$ is RN-operator between Banach spaces $\hat{E_B}$ and $\hat{F_V}$.*

Here, Ψ_V and ϕ_B are the natural maps (cf. definitions in Section 1).

Remark 13 *For the operators in Banach spaces our definition of RN-operators coincides with the original definitions from* [9], [6]. *This must be clear.*

Remark 14 *The usual definition of a weakly compact operator between locally convex spaces is: $T : E \to F$ is weakly compact if T maps a neighborhood of zero in E to a relatively weakly compact subset of F. The analogous definition can be given for the compact operators. For the weakly compact case this means that*

(1) There exists an absolutely convex neighborhood $V = V(0)$ in E such that if $D := T(V)$, then D is bounded in F and the natural injection $\phi_D : F_D \to F$ is weakly compact.

Since every weakly compact operator in Banach spaces is Radon-Nikodym, we get from the definition that every weakly compact operator in l.c.v.s. is an RN-operator.

Following this way, we can defined also a class of "bounded RN-operators". Namely, let T maps E to F. T is a bounded RN-operator if T takes a neighborhood V to a bounded subset of F and the natural map $\phi_{T(V)} : F_{T(V)} \to F$ is "Radon-Nikodym". But how to understand "Radon-Nikodym" in this case where an operator maps a normed (or Banach) space to a locally convex space? We can go by a geometrical way (saying that the image of $\phi_{T(V)}$ is subset s-dentable. Here the map $\phi_{T(V)}$ is one-to-one, and we can follow the assertion from [8] for operators in Banach spaces: if $U : X \to Y$ is one-to-one then U is RN iff the U-image of the unit ball is subset s-dentable). Another way is just to apply the main definition from this paper.

Or, we can go by a traditional way: for $T : X \to F$ with X Banach and F locally convex, say that T is of type RN if for every operator U from an L_1-space to X the composition TU admits an integral representation with a (strongly) integrable F-valued function. But in this case we are to give a good definition of the "integrability" of an F-valued function. All these are the topic for the further considerations in another paper.

Let us mention that in every "right" definition of RN-operator there must be an "ideal property": ATB is RN for all linear continuous A, B if T is RN. Thus, if we apply the definition 12 above as the main definition (in this paper), then every operator "of type RN" considered above in this remark is "right" Radon-Nikodym. This is one of the reason that here we will deal only with Definition 12.

If $E = X$ is a Banach space then $T \in RN(X, F)$ iff for every (a.c.) neighborhood $V = V(0) \subset F$ the composition $\Psi_V T$ belongs to $RN(X, \widehat{F}_V)$. If F is a Banach space too, then, evidently, $T \in RN$ in the sense of Definition 12 iff it is an RN-operator in the usual sense of [9] and [6]. If E is a l.c.v.s. and $F = Y$ is Banach, then $T \in RN(E, Y)$ iff for every bounded complete absolutely convex subset $B \subset E$ the natural map $T\phi_B : E_B \to E \to Y$ is Radon-Nikodym, what means that $T \in RN(E, Y)$ iff for any finite measure space (Ω, Σ, μ) and for every μ-continuous E-valued measure $\bar{m} : \Sigma \to E$ with bounded μ-average $\{\frac{\bar{m}(A)}{\mu(A)} : A \in \Sigma, \mu(A) \neq 0\}$ the measure $T\bar{m} : \Sigma \to Y$ has a Bochner derivative with respect to μ. From this it follows that an operator T between locally convex spaces E, F is Radon-Nikodym (in the sense of Definition 12) iff for any finite measure space (Ω, Σ, μ) and for every μ-continuous E-valued measure $\bar{m} : \Sigma \to E$ with bounded μ-average, for every a.c. neighborhood $V = V(0) \subset F$ the measure $\Psi_V T\bar{m} : \Sigma \to \widehat{F}_V$ has a Bochner derivative with respect to μ.

Proposition 15 *Let $\Psi_V : E \to \hat{E_V}$, where $V = V(0)$ is an absolutely convex open neighborhood of zero then ${\Psi_V}^{-1}(D_1(0)) = V$, where $D_1(0)$ is open unit ball in $\hat{E_V}$.*

Proof: " $\supseteq$ ": Let $x \in V$, then by definition of $\|.\|_{\hat{E_V}}$, $\|\Psi_V(x)\| < 1$ so $V \subseteq {\Psi_V}^{-1}(D_1(0))$

" $\subseteq$ ": Let $x \in {\Psi_V}^{-1}(D_1(0))$; then $\Psi_V(x) \in D_1(0)$. This implies $\|\Psi_V(x)\| = 1 - c < 1$ for some $c > 0$, so $\inf\{\lambda > 0 : x \in \lambda V\} = 1 - c < 1$. This implies $x \in (1 - \frac{c}{2})V$. Since V is absolutely convex so $\lambda V \subseteq V$ for every $|\lambda| < 1$, therefore $x \in V$. Hence ${\Psi_V}^{-1}(D_1(0)) = V$. □

Theorem 16 *Let E, F be locally convex vector spaces and $T : E \to F$ be a linear continuous operator. Consider the following conditions:*

1) $T \in RN(E, F)$.

2) Each bounded set in E is T-dentable.

3) Each bounded set in E is T-s-dentable.

4) Each bounded set in E is T-f-dentable

We have $1) \Leftrightarrow 4)$, $2) \Rightarrow 3) \Rightarrow 4)$. If the space E is an SBM-space then all the conditions are equivalent.

Proof: $1) \Rightarrow 4)$. Let $B_0 \subseteq E$ be bounded and $V \subseteq F$ be an absolutely convex neighborhood of 0; put $B = \overline{\Gamma(B_0)}$. By assumption in $1)$, the composition operator $\Psi_V \circ T \circ \phi_B : E_B \to E \to F \to \hat{F_V}$ is an RN-operator from E_B to $\hat{F_V}$. By theorem 7 each bounded subset in E_B is $\Psi_V T \phi_B$-f-dentable. In particular, B_0 is $\Psi_V T \phi_B$-f-dentable. Let $\epsilon > 0$, there exists $x \in B_0$ such that

$$(i)\ x \notin co\,(B_0 \setminus (\Psi_V T \phi_B)^{-1}(D_\epsilon(\Psi_V T \phi_B(x)))).$$

Proposition 15 implies that ${\Psi_V}^{-1}(D_\epsilon(\Psi_V T \phi_B(x))) = {\Psi_V}^{-1}(\Psi_V(T(x))) + \epsilon V$, so we get

$$(ii)\ x \notin co\,(B_0 \setminus T^{-1}(T(x) + \epsilon V),$$

otherwise there exist x_1, x_2 ..., x_n in B_0 where $x_k \notin T^{-1}(T(x) + \epsilon V)$ and λ_1, λ_2 ...,λ_n where $0 \leq \lambda_k \leq 1$ and $\sum_{k=1}^n \lambda_k = 1$ such that $\sum_{k=1}^n \lambda_k x_k = x$. Since $x_k \notin T^{-1}(T(x) + \epsilon V)$ so $T(x_k) \notin (T(x) + \epsilon V)$ hence $T(x_k) \notin {\Psi_V}^{-1}(\Psi_V(T(x)))$ for each $k = 1, 2$ ... n. So, we get $x_k \notin {\phi_B}^{-1} T^{-1} {\Psi_V}^{-1}(D_\epsilon(\Psi_V T \phi_B(x)))$ and $\sum_{k=1}^n \lambda_k x_k = x$ which is contradicts (i), therefore (ii) holds and hence B_0 is T-V-f-dentable, which proofs $4)$.

$4) \Rightarrow 1)$. Let $B \subseteq E$ be a bounded, absolutely convex and complete subset and $V \subseteq F$ be an absolutely convex neighborhood of 0. We need to show that the operator $\Psi_V \circ T \circ \phi_B : E_B \to E \to F \to \hat{F_V}$ is RN. By theorem 7, it is enough to show that each bounded B_0 is $\Psi_V T \phi_B$-f-dentable. Let $\epsilon > 0$; by assumption in $4)$ there exists an $x \in B_0$ such that

$$x \notin co\,(B_0 \setminus T^{-1}(T(x) + \epsilon V))\,),$$

which, by above proposition, gives that

$$x \notin co\,(B_0 \setminus (\Psi_V T \phi_B)^{-1}(D_\epsilon(\Psi_V T \phi_B(x)))\,).$$

This proofs $1)$.

$2) \Rightarrow 3) \Rightarrow 4)$. Follows from definitions.

Suppose now that E is an SBM-space.

$4) \Rightarrow 2)$. Let $B \subseteq E$ be closed bounded and convex and let V be an absolutely convex neighborhood

of zero in F. We show that B is subset T-V-dentable. Clearly, $B_0 \subset B$ is T-V-f-dentable if and only if B_0 is $U = T^{-1}(V)$-f-dentable. From assumption, it follows that B is subset U-f-dentable. Applying the consequence of theorem 8, we get that B is subset U-dentable, or B is subset T-V-dentable, which proofs 2). □

Example 17 Consider any uncountable set Γ and the classical Banach spaces $c_0(\Gamma)$ and $l_1(\Gamma)$; $c_0(\Gamma)^* = l_1(\Gamma)$. The closed unit ball B of $l_1(\Gamma)$ is weak* compact and an Eberlein compact in this weak* topology (see, e.g., [2]). This implies that every separable subset of (B, w^*) is metrizable and that (B, w^*) has the Frechet-Uryson property, i.e. every point in the closure of a subset is a limit of a sequence of this subset. Thus, the space $(l_1(\Gamma), w^*)$ is of type SBM. The Banach space $l_1(\Gamma)$ has the RNP. Therefore, every bounded subset of the space is dentable (s-dentable, f-dentable), and thus every bounded subset of $(l_1(\Gamma), w^*)$ is (w^*)-dentable too. The identity map $(l_1(\Gamma), w^*) \to (l_1(\Gamma, w^*))$ is Radon-Nikodym in the sense of Definition 12. Moreover the space has the RNP in the sense of the paper [11]. This directly follows by Theorem 16, but not *directly* from the results of [11]. On the other case, for $l_1(\Gamma)$ this fact is trivial. We can also say (by the same considerations) that if X is any weakly generated Banach space (see [2]) with $X^* \in RN$, then the space $(X^*, \sigma(X^*, X))$ has all just mentioned properties (clearly, or by application of Theorem 16).

Finally, if X is WCG, then $(X^*, \sigma(X^*, X))$ is SBM. So, the theorem can be applied for all such spaces.

On the other hand, there is an example (see [4], Theorem 4), which shows that there exists a separable Banach space Y such that the space $Y_{\sigma:=}(Y, \sigma(Y, Y^*))$ does not have the RNP (in the sense of [11]) and every bounded subset of which (of Y_σ) is s-dentable. Thus, the above theorem is not true for general l.c.v.s.

Let us note that a l.c.v.s. E has the RNP in sense of [11] iff the identity map $E \to E$ is an RN-operator. Therefore, we obtain a generalization of a theorem from [11]:

Corollary 18 *For every SBM-space E (in particular, for every quasi complete l.c.v.s. with metrizable bounded subsets, or for every Frechet space) the following are equivalent:*

1. E has the RNP of [11].
2. Each bounded set in E is dentable.
3. Each bounded set in E is s-dentable.
4. Each bounded set in E is f-dentable.

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