Some remarks on spectra of nuclear operators

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Abstract. We give criteria for the spectra of some nuclear operators in subspaces of quotients of L_p-spaces to be central-symmetric, as well as for the spectra of linear operators in Banach spaces to be Z_d-symmetric in the sense of B. Mityagin. Also, we present a short proof of a generalization of Mityagin's theorem.

1. Introduction. It was shown by M. I. Zelikin in [16] that the spectrum of a nuclear operator in a separable Hilbert space is central-symmetric iff the spectral traces of all odd powers of the operator equal zero. Recall that the spectrum of every nuclear operator in a Hilbert space consists of non-zero eigenvalues of finite algebraic multiplicity, which have no limit point except possibly zero, and maybe zero. This system of all eigenvalues (written according to their multiplicities) is absolutely summable, and the spectral trace of any nuclear operator is, by definition, the sum of all its eigenvalues (taken according their multiplicities).

The space of nuclear operators in a Hilbert space may be defined as the space of all trace-class operators (see [14, c. 77]); in this case we say about the "nuclear trace" of an operator). Trace-class operators in a Hilbert space can be considered also as the elements of the completion of the tensor product of the Hilbert space and its Banach dual with respect to the greatest crossnorm on this tensor product [14, c. 119]. The well known Lidskiĭ theorem [5] says that the nuclear trace of any nuclear operator in a Hilbert space (or, what is the same, of the corresponding tensor element) coincides with its spectral trace. Thus, Zelikin's theorem [16] can be reformulated in the following way: the spectrum of a nuclear operator in a separable Hilbert space is central-symmetric iff the nuclear traces of all odd powers of the corresponding tensor element are zero.

One of the aim of our notes is to give an exact generalization of this result to the case of tensor elements of so-called s-projective tensor products of subspaces of quotients of $L_p(\mu)$-spaces. In particular, we get as a consequence Zelikin's theorem (taking $p = 2$).

Another problem which is under consideration in our notes is concentrated around the so called $\mathbb{Z}_d$-symmetry of the spectra of the linear operators. The notion of $\mathbb{Z}_d$-symmetry of the spectra was introduced by B.S. Mityagin in a preprint [6] and in his paper [7]. He is interested there in a generalization of the result from [16] in two directions: to extend Zelikin's theorem to the case of general Banach spaces and to change the property of a compact operator to have central-symmetric spectrum to the property of the one to have $\mathbb{Z}_d$-symmetric spectrum. Roughly speaking, $\mathbb{Z}_d$-symmetry of a spectrum of a compact operator T means that for any non-zero eigenvalue λ of T the spectrum contains also as eigenvalues of the same algebraic multiplicities all "d-shifted" numbers $t\lambda$ for $t \in \sqrt[d]{1}$. B.S. Mityagin has obtained a very nice result, showing that the spectrum of a compact operator T in an arbitrary Banach space, some power T^m of which is nuclear, is $\mathbb{Z}_d$-symmetric iff for all large enough integers of type $kd + r$ $(0 < r < d)$ the nuclear traces of T^{kd+r} are zero. We present some thoughts around this theorem, giving, in particular, a short (but not so elementary as in [6, 7]) proof for the case where the operator is not necessarily compact. Let us mention, however, that the proof from [6, 7] can be adapted for this situation too.

Some words about the content of the paper.

In Section 2, we introduce some notations, definitions and terminology in connection with so called s-projective tensor products, s-nuclear operators and the approximation properties of order s, $s \in (0, 1]$. We formulate here two auxiliary assertions from the author's paper [13]; they give us generalized Grothendieck-Lidskiĭ trace formulas which will be useful in the next section.

Section 3 contains an exact generalization of Zelikin's theorem. In this section we present a criterium for the spectra of s-nuclear operators in subspaces of quotients of L_p-spaces to be central-symmetric.

Results of Section 4 show that the criterium of the central symmetry, obtained in the previous section, is optimal. In particular, we present here (Theorem 2) sharp examples of s-nuclear operators T in the spaces l_p, $1 \le p \le +\infty$, $p \ne 2$, for which trace $T = 1$ and $T^2 = 0$.

Finally, Section 5 is devoted to the study of Mityagin's $\mathbb{Z}_d$-symmetry of the spectra of linear operators. Our aim here is to give a short (but using Fredholm Theory) proof of Mityagin's theorem [6-7] for arbitrary linear continuous (Riesz) operators. Firstly, we consider a $\mathbb{Z}_2$ situation (central symmetry) to clarify an idea which is to be used then in the general case. We finish the paper with a short proof of the theorem from [6-7] for continuous (not necessarily compact) operators and with some simple examples of applications.

2. Preliminaries. We denote by $X, Y, \dots$ Banach spaces, $L(X, Y)$ is a Banach space of all linear continuous operators from X to Y; $L(X) := L(X, X)$. For a Banach dual to a space X we use the notation X^*. If $x \in X$ and $x' \in X^*$, then $\langle x', x\rangle$ denotes the value $x'(x)$.

By $X^*\widehat{\otimes}X$ we denote the projective tensor product of the spaces X^* and X [2] (see also [11, 12]). It is a completion of the algebraic tensor product $X^* \otimes X$ (considered as a linear space of all finite rank continuous operators w in X) with respect to the norm

$$||w||_\wedge := \inf\{\left(\sum_{k=1}^N ||x'_k||\,||x_k||\right) : \ w = \sum_{k=1}^N x'_k \otimes x_k\}.$$

Every element u of the projective tensor product $X^*\widehat{\otimes}X$ can be represented in the form

$$u = \sum_i \lambda_i x'_i \otimes x_i,$$

where $(\lambda_i) \in l_1$ и $||x'_i|| \le 1$, $||x_i|| \le 1$ [2].

More generally, if $0 < s \le 1$, then $X^*\widehat{\otimes}_s X$ is a subspace of the projective tensor product, consisting of the tensor elements $u, u \in X^*\widehat{\otimes}X$, which admit representations of the form $u = \sum_{k=1}^\infty x'_k \otimes x_k$, where $(x'_k) \subset X^*, (x_k) \subset X$ and $\sum_{k=1}^\infty ||x'_k||^s\, ||x_k||^s < \infty$ [2, 11, 12]. Thus, $X^*\widehat{\otimes}X = X^*\widehat{\otimes}_1 X$.

On the linear space $X^* \otimes X$, a linear functional "trace" is defined by a natural way. It is continuous on the normed space $(X^* \otimes X, ||\cdot||_\wedge)$ and has the unique continuous extension to the space $X^*\widehat{\otimes}X$, which we denote by trace.

Every tensor element $u, u \in X^*\widehat{\otimes}X$, of the form $u = \sum_{k=1}^\infty x'_k \otimes x_k$ generates naturally an operator $\widetilde{u} : X \to X$, $\widetilde{u}(x) := \sum_{k=1}^\infty \langle x'_k, x\rangle\, x_k$ for $x \in X$. This defines a natural mapping $j_1 : X^*\widehat{\otimes}X \to L(X)$. The operators, lying in the image of this map, are called nuclear [2], [8]. More generally, if $0 < s \le 1$, $u = \sum_{k=1}^\infty x'_k \otimes x_k$ and $\sum_{k=1}^\infty ||x'_k||^s\, ||x_k||^s < \infty$, then the corresponding operator $\widetilde{u}$ is called s-nuclear [11, 12]. By j_s we denote a natural map from $X^*\widehat{\otimes}_s X$ to $L(X)$. We say that a space X has the approximation property of order s, $0 < s \le 1$ (the AP_s), if the canonical mapping j_s is one-to-one [11, 12]. Note that the AP_1 is exactly the approximation property AP of A. Grothendieck [2], [8]. Classical spaces, such as $L_p(\mu)$ and $C(K)$, have the approximation property. If a space X has the AP_s, then we can identify the tensor product $X^*\widehat{\otimes}_s X$ with the space $N_s(X)$ of all s-nuclear operators in в X (i.e. with the image of this tensor product under the map j_s). In this case for every operator $T \in N_s(X) = X^*\widehat{\otimes}_s X$ the functional trace T is well defined and called the *nuclear trace* of the operator T.

It is clear that if a Banach space has the approximation property, then it has all the properties AP_s, $s \in (0, 1]$. Every Banach space has the property $AP_{2/3}$ (A. Grothendiek [2], see also [10]). Since each Banach space is a subspace of an $L_\infty(\mu)$-space, the following fact (to be used below) is a generalization of the mentioned result of A. Grothendieck:

Lemma 1 [13, Corollary 10]. Let $s \in (0, 1], p \in [1, \infty]$ and $1/s = 1 + |1/p - 1/2|$. If a Banach space Y is isomorphic to a subspace of a quotient (or to a quotient of a subspace) of some $L_p(\mu)$-space, then it has the AP_s.

Thus, for such spaces we have an equality $Y^*\widehat{\otimes}_s Y = N_s(Y)$ and the nuclear trace of any operator $T \in N_s(Y)$ is well defined.

We will need also the following auxiliary assertion (the first part of which is a consequence of the previous lemma).

Lemma 2 [13, Theorem 1]. Let Y is a subspace of a quotient (or to a quotient of a subspace) of some $L_p(\mu)$-space, $1 \le p \le \infty$. If $T \in N_s(Y)$, where $1/s = 1 + |1/2 - 1/p|$, then

1. the nuclear trace of the operator T is well defined,

2. $\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, where $\{\lambda_n(T)\}$ is a system of all eigenvalues of the operator T (written according their algebraic multiplicities) and

$$\text{trace } T = \sum_{n=1}^{\infty} \lambda_n(T).$$

Following [16], we say that a spectrum of a compact operator in a Banach space is *central-symmetric*, if for every its eigenvalue λ the number $-\lambda$ is also its eigenvalue and of the same algebraic multiplicity. We shall use the same terminology in the case of operators all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero; the corresponding eigenvalue sequence for such an operator T will be denoted by $\operatorname{sp}(T)$; thus it is an unordered sequence of all eigenvalues of T taken according their multiplicities.

3. On central symmetry. Let us note firstly that the theorem of Zelikin (in the form as it was formulated in [16]) can not be extended to the case of general Banach spaces, even if the spaces have the Grothendieck approximation property.

Example 1. Let U be a nuclear operator in the space l_1, constructed in [8, Proposition 10.4.8]. This operator has the property that trace $U = 1$ and $U^2 = 0$. Evidently, the spectrum of this operator is $\{0\}$. Let us note that the operator is not only nuclear, but also belongs to the space $N_s(l_1)$ for all $s \in (2/3, 1]$. It is not possible to present such an example in the case of 2/3-nuclear operators (see Corollary 3 below). Note also that, however, the traces of all operators $U^m, m = 2, 3, \dots,$ (in particular, U^{2n-1}) are equal to zero.

Remark 1. For every nuclear operator $T : X \to X$ and for any natural number $n > 1$, the nuclear trace trace T^n is well defined (see [2, Chap II, §1, n°4, Corollaire 2, p. 16]) and equal the sum of all its eigenvalues (according their multiplicity) [2, Chap II, §1, n°4, Corollaire 1, p. 15]. Therefore, if the spectrum of a nuclear operator $T : X \to X$ is central-symmetric, then for each odd $m = 3, 5, 7, \dots$ the nuclear trace of the operator T^m is equal to zero. This follows from the fact that the eigenvalue sequences of T and T^m can be arranged in such a way that $\{\lambda_n(T)^m\} = \{\lambda_n(T^m)\}$ (see, e.g., [9, 3.2.24, p. 147]).

Let us formulate and prove the central result of this section.

Theorem 1.Let Y be a subspace of a quotient (or a quotient of a subspace) of an L_p-space, $1 \le p \le \infty$, and $u \in Y^*\widehat{\otimes}_s Y$, where $1/s = 1 + |1/2 - 1/p|$, The spectrum of the operator $\widetilde{u}$ is central-symmetric iff trace $u^{2n-1} = 0, n = 1, 2, \dots$.

Proof. If the spectrum of $\widetilde{u}$ is central-symmetric, then, by Lemma 2, trace $u =$ trace $\widetilde{u} = \sum_{n=1}^{\infty} \lambda_n(T) = 0$; also, by Remark 1, trace $u^m = \sum_{n=1}^{\infty} \lambda_n(T^m) = 0$ for $m = 3, 5, \dots$.

To prove the inverse, we need some information from Fredholm Theory. Let u be an element of the projective tensor product $X^*\widehat{\otimes}X$, where X is an arbitrary Banach space. Recall that the Fredholm determinant $\det(1 - zu)$ of u (see [2, Chap II, §1, n°4, p. 13] or [3], [8], [9]) is an entire function

$$\det(1 - zu) = 1 - z \text{ trace } u + \dots + (-1)^n z^n \alpha_n(u) + \dots,$$

all zeros of which are exactly (according to their multiplicities) the inverses of nonzero eigenvalues of the operator $\widetilde{u}$, associated with the tensor element u. By [2, Chap II, §1, $n°4$, Corollaire 2, pp. 17-18], this entire function is of the form

$$\det(1 - zu) = e^{-z \operatorname{trace} u} \prod_{i=1}^{\infty} (1 - zz_i)\, e^{zz_i},$$

where $\{z_i = \lambda_i(\widetilde{u})\}$ is a system of all eigenvalues of the operator $\widetilde{u}$ (written according their algebraic multiplicities). Also, there exists a $\delta > 0$ such that for all $z, |z| \le \delta$, we have

$$\det(1 - zu) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} z^n \operatorname{trace} u^n\right) \tag{1}$$

(see [3, p. 350]; cf. [1, Theorem I.3.3, p. 10]).

Now, let $u \in Y^* \widehat{\otimes}_s Y$ be as in the formulation of our theorem and suppose that trace $u^{2n-1} = 0, n = 1, 2, \ldots$. By (**??**), we get: for a neighborhood $U = U(0)$ of zero in $\mathbb{C}$, $\det(1 - zu) = \det(1 + zu)$ for $z \in U$. Therefore, the entire function $\det(1 - zu)$ is even. By definition of $\det(1 - zu)$, the sequence of zeros of this function is exactly the eigenvalue sequence of $\widetilde{u}$. Hence, the spectrum of $\widetilde{u}$ is central-symmetric.

Since under the conditions of Theorem 1 the space Y has the AP_s, the tensor product $Y^* \widehat{\otimes}_s Y$ can be identified naturally with the space of all s-nuclear operators in Y. Hence, the statement of Theorem 1 may be reformulated in the following way:

Corollary 1. *Let $s \in [2/3, 1], p \in [2, \infty], 1/s = 1 + |1/2 - 1/p|$, Y be a subspace of a quotient (or a quotient of a subspace) of an L_p-space, T be an s-nuclear operator in Y. The spectrum of T is central-symmetric iff* trace $T^{2n-1} = 0$, $n \in \mathbf{N}$.

Corollary 2 [16]. *The spectrum of a nuclear operator T, acting on a Hilbert space, is central-symmetric iff* trace $T^{2n-1} = 0$, $n \in \mathbf{N}$.

For a proof, it is enough to apply Theorem 1 for the case $p = 2$.

Corollary 3. *The spectrum of a 2/3-nuclear operator T, acting on an arbitrary Banach space, is central-symmetric iff* trace $T^{2n-1} = 0$, $n \in \mathbf{N}$.

For a proof, it is enough to apply Theorem 1 for the case $p = \infty$, taking into account the fact that every Banach space is isometric to a subspace of an $L_\infty(\mu)$-space.

In connection with Corollary 3, let us pay attention again to the nuclear operator from Example 1.

4. Sharpness of results of Section 3. Now we will show that the statement of Theorem 1 is sharp and that the exponent s is optimal if p is fixed (if of course $p \ne 2$, i.e. $s \ne 1$).

Consider the case $2 < p \le \infty$. In this case $1/s = 1 + |1/2 - 1/p| = 3/2 - 1/p$. In a paper of the author [12, Example 2] the following result was obtained (see a proof in [12]):

($\star$) *Let $r \in [2/3, 1), p \in (2, \infty], 1/r = 3/2 - 1/p$. There exist a subspace Y_p of the space l_p (c_0 if $p = \infty$) and a tensor element $w_p \in Y_p^* \widehat{\otimes}_1 Y_p$ such that $w_p \in Y_p^* \widehat{\otimes}_s Y_p$ for every $s > r$,* trace $w_p = 1, \tilde{w}_p = 0$ *and the space Y_p (as well as Y_p^*) has the AP_r (but evidently does not have the AP_s if $1 \geq s > r$). Moreover, this element admits a nuclear representation of the form*

$$w_p = \sum_{k=1}^{\infty} \mu_k \, x'_k \otimes x_k, \quad where \ \ ||x'_k|| = ||x_k|| = 1, \sum_{k=1}^{\infty} |\mu_k|^s < \infty \; \forall \, s > r.$$

Evidently, we have for a tensor element $u := w_p$ from the assertion ($\star$) : trace $u = 1$ and the spectrum of the operator $\widetilde{u}$ equals $\{0\}$.

The case where $2 < p \leq \infty$ can be considered analogously (with an application of the assertion ($\star$) to a "transposed" tensor element $w_p^t \in Y_p \widehat{\otimes}_1 Y_p^*$.)

As was noted above (Example 1), there exists a nuclear operator U in l_1 such that $U^2 = 0$ and trace $U = 1$. The following theorem is an essential generalization of this result and gives us the sharpness of the statement of Corollary 1 (even in the case where $Y = l_p$).

Theorem 2. Let $p \in [1, \infty], p \neq 2$, $1/r = 1 + |1/2 - 1/p|$. There exists a nuclear operator V in l_p (in c_0 if $p = \infty$) such that

1) $V \in N_s(l_p)$ for each $s \in (r, 1]$;
2) $V \notin N_r(l_p)$;
3) trace $V = 1$ and $V^2 = 0$.

Proof. Suppose that $p > 2$. Consider the tensor element $w := w_p$ from the assertion ($\star$) and its representation $w = \sum_{k=1}^{\infty} \mu_k \, x'_k \otimes x_k$, where $||x'_k|| = ||x_k|| = 1$ и $\sum_{k=1}^{\infty} |\mu_k|^s < \infty$ for each $s, s > r$. Let $l : Y := Y_p \to l_p$ be the identity inclusion. Let y'_k be an extension of the functional x'_k $(k = 1, 2, \dots)$ from the subspace Y to the whole space l_p with the same norm and set $v := \sum_{k=1}^{\infty} \mu_k \, y'_k \otimes l(x_k)$. Then $v \in l_{p'} \widehat{\otimes}_s l_p$ $(1/p + 1/p' = 1)$ for each $s \in (r, 1]$, trace $v = \sum \mu_k \, \langle y'_k, l(x_k) \rangle = 1$ and $\widetilde{v}(l_p) \subset l(Y) \subset l_p$. On the other hand, we have a diagram:

$$Y \xrightarrow{l} l_p \xrightarrow{\widetilde{v}_0} Y \xrightarrow{l} l_p \xrightarrow{\widetilde{v}_0} Y \xrightarrow{\widetilde{v}_0} l_p,$$

where $\widetilde{v}_0$ is an operator generated by $\widetilde{v}$, $\widetilde{v} = l\widetilde{v}_0$ and $\widetilde{v}_0 l = \widetilde{w} = 0$. Put $V := \widetilde{v}$. Clearly, trace $V = 1$ and the spectrum sp $V^2 = \{0\}$. Let us note that $V \notin N_r(l_p)$ (by Lemma 2). If $p \in [1, 2)$, then it is enough to consider the adjoint operator.

It follows from Theorem 2 that the assertion of Corollary 1 is optimal already in the case of the space $Y = l_p$ (which is, by the way, has the Grothendieck's approximation property).

5. Generalizations: around Mityagin's theorem. Recall that if $T \in L(X)$ and, for some $m \in \mathbb{N}$, T^m is a Riesz operator (see, e.g., [4, p. 943] for a definition), then T is a Riesz operator too (see, e.g., [9, 3.2.24, p. 147]). In particular, if T^m is compact, then all non-zero spectral values $\lambda(T) \in \mathrm{sp}\,(T)$ are eigenvalues of finite (algebraic) multiplicity and have no limit point except possibly zero. Also, in this case the eigenvalue sequences of T and T^m can be arranged in such a way that

$\{\lambda_n(T)^m\} = \{\lambda_n(T^m)\}$ (see [9, 3.2.24, p. 147]). Recall that in this case we denote by $\mathrm{sp}\,(T)$ (resp., by $\mathrm{sp}\,(T^m)$ the sequence $\{\lambda_n(T)^m\}$ (resp., $\{\lambda_n(T^m)\}$).

We are going to present a short proof of the theorem of B. Mityagin from [6, 7]. To clarify our idea of the proof, let us consider firstly the simplest case where $d = 2$.

Theorem 3. Let X be a Banach space and $T \in L(X)$. Suppose that some power of T is nuclear. The spectrum of T is central-symmetric iff there is an integer $K \geq 0$ such that for every $l > K$ the value trace T^l is well defined and trace $T^{2l+1} = 0$ for all $l > K$.

Proof. Suppose that $T \in L(X)$ and there is an $M \in \mathbb{N}$ so that $T^M \in N(X)$. Fix an odd $N_0, N_0 > M$, with the property that $T^{N_0} \in N_{2/3}(X)$ (it is possible since a product of three nuclear operators is 2/3-nuclear) and trace $T^{N_0+2k} = 0$ for all $k = 0, 1, 2, \dots$. By Corollary 3, the spectra of all T^{N_0+2k} are central-symmetric (since, e.g., trace T^{N_0} = trace $(T^{N_0})^3$ = trace $(T^{N_0})^5 = \dots = 0$ by assumption). Assume that the spectrum of T is not central-symmetric. Then there exists an eigenvalue $\lambda_0 \in \mathrm{sp}\,(T)$ such that $-\lambda_0 \notin \mathrm{sp}\,(T)$.

Now, $\lambda_0^{N_0} \in \mathrm{sp}\,(T^{N_0})$, so $-\lambda_0^{N_0} \in \mathrm{sp}\,(T^{N_0})$. Hence, there exist $\mu_{N_0} \in \mathrm{sp}\,(T)$ and θ_{N_0} so that $|\theta_{N_0}| = 1$, $\mu_{N_0}^{N_0} = -\lambda_0^{N_0}$ and $\mu_{N_0} = \theta_{N_0}\lambda_0$, $\theta_{N_0} \neq -1$. Analogously, $\lambda_0^{N_0+2} \in \mathrm{sp}\,(T^{N_0+2})$, so $-\lambda_0^{N_0+2} \in \mathrm{sp}\,(T^{N_0+2})$. Hence, there exist $\mu_{N_0+2} \in \mathrm{sp}\,(T)$ and θ_{N_0+2} so that $|\theta_{N_0+2}| = 1$, $\mu_{N_0+2}^{N_0+2} = -\lambda_0^{N_0+2}$ and $\mu_{N_0+2} = \theta_{N_0+2}\lambda_0$, $\theta_{N_0+2} \neq -1$ etc. By induction we get the sequences $\{\mu_{N_0+2k}\}_{k=0}^{\infty}$ and $\{\theta_{N_0+2k}\}_{k=0}^{\infty}$ with the properties that $\mu_{N_0+2k} \in \mathrm{sp}\,(T)$, $|\theta_{N_0+2k}| = 1$, $\mu_{N_0+2k}^{N_0+2k} = -\lambda_0^{N_0+2k}$ and $\mu_{N_0+2k} = \theta_{N_0+2k}\lambda_0$, $\theta_{N_0+2k} \neq -1$. Since $\mu_{N_0+2k} \in \mathrm{sp}\,(T)$ and $|\mu_{N_0+2k}| = |\lambda_0| > 0$, the sequence $\{\mu_{N_0+2k}\}$ is finite as a set, i.e., we have that $\mu_{N_0+2K} = \mu_{N_0+2K+2} = \dots$ for some $K > 1$. It follows that $\theta_{N_0+2K} = \theta_{N_0+2K+2} = \dots$. But $\theta_{N_0+2k}^{N_0+2k} = -1$ for all k. Thus $\theta_{N_0+2K}^{l} = -1$ for every odd $l \geq N_0 + 2K$. Therefore, $\theta_{N_0+2K} = -1$. Contradiction

Now we are going to consider a general case of a notion of $\mathbb{Z}_d$-symmetry of a spectra, introduced and investigated by B. Mityagin in [6, 7]. Let T be an operator in X, all non-zero spectral values of which are eigenvalues of finite multiplicity and have no limit point except possibly zero. Recall that we denote by $\mathrm{sp}\,(T)$ the corresponding unordered eigenvalue sequence for T (possibly, including zero). For a fixed $d = 2, 3, \dots$ and for the operator T, the spectrum of T is called $\mathbb{Z}_d$-symmetric, if $\lambda \in \mathrm{sp}\,(T)$ implies $t\lambda \in \mathrm{sp}\,(T)$ for every $t \in \sqrt[d]{1}$.

Let $r \in (0, \infty]$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < r\}$, $f : \mathbb{D} \to \mathbb{C}$. and $d \in \mathbb{N} \setminus \{1\}$. We say that f is d-even if $f(tz) = f(z)$ for every $t \in \sqrt[d]{1}$.

Lemma 3. Let $\Phi(X)$ be a linear subspace of $X^* \widehat{\otimes} X$ of spectral type l_1, i.e., for every $v \in \Phi(X)$ a series $\sum_{\lambda \in \mathrm{sp}\,(\widetilde{v})} |\lambda|$ is convergent. Let $d \in \mathbb{N}, d > 1$. If $u \in \Phi(X)$, then the Fredholm determinant $\det\,(1 - zu)$ is d-even iff the eigenvalue sequence of $\widetilde{u}$ is $\mathbb{Z}_d$-symmetric iff trace $u^{kd+r} = 0$ for all $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots, d - 1$.

Proof. If the function $\det\,(1 - zu)$ is d-even, then the eigenvalue sequence of $\widetilde{u}$ is $\mathbb{Z}_d$-symmetric, since this sequence coincides with the the sequence of all zeros of $\det\,(1 - zu)$ (according their multiplicities).

If the eigenvalue sequence of $\widetilde{u}$ is $\mathbb{Z}_d$-symmetric, then trace $u = \sum_{\lambda\in \mathrm{sp}\,(\widetilde{u})} \lambda = 0$ (since $\Phi(X)$ is of spectral type l_1 and $\sum_{t\in \sqrt[d]{1}} t = 0$). Also, by the same reason trace $u^{kd+r} = 0$ for all $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots, d-1$, since the spectrum of $\widetilde{u}^l$ is absolutely summable for every $l \geq 2$ and we may assume that $\{\lambda_m(\widetilde{u}^l)\} = \{\lambda_m(\widetilde{u})^l\}$ (for every fixed l) [9, 3.2.24, p. 147].

Now, let trace $u^{kd+r} = 0$ for all $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots, d-1$. By (**??**), $\det(1 - zu) = \exp(-\sum_{m=1}^{\infty} \frac{1}{md} z^{md}\,\mathrm{trace}\, u^{md})$ in a neighborhood U of zero. Therefore, this function is d-even in the neighborhood U. By the uniqueness theorem $\det(1 - zu)$ is d-even in $\mathbb{C}$.

Corollary 4. For any Banach space X and for every $u \in X^*\widehat{\otimes}_{2/3}X$ the conclusion of Lemma 3 holds.

Corollary 5. Let Y be a subspace of a quotient of an L_p-space, $1 \leq p \leq \infty$. For any $u \in Y^*\widehat{\otimes}_s Y$, where $1/s = 1 + |1/2 - 1/p|$, the conclusion of Lemma 3 holds.

Now we are ready to present a short proof of the theorem of B. Mityagin [6, 7]. Note that the theorem in [6, 7] is formulated and proved for compact operators, but the proof from [6, 7] can be easily adapted for the general case of linear operators.

Theorem 4. Let X be a Banach space and $T \in L(X)$. Suppose that some power of T is nuclear. The spectrum of T is $\mathbb{Z}_d$-symmetric iff there is an integer $K \geq 0$ such that for every $l > Kd$ the value trace T^l is well defined and trace $T^{kd+r} = 0$ for all $k = K, K+1, K+2, \dots$ and $r = 1, 2, \dots, d-1$.

Proof. Fix $N_0 \in \mathbb{N}$ such that T^{N_0} is 2/3-nuclear (it is possible by a composition theorem from [2, Chap II, §1, $n°3$, Theorem 3, p. 10]). Note that, by A. Grothendieck, the trace of T^l is well defined for all $l \geq N_0$.

Suppose that the spectrum of T is $\mathbb{Z}_d$-symmetric. Take an integer $l := kd + r \geq N_0$ with $0 < r < d$. Since the spectrum of T^l is absolutely summable, trace $T^l = \sum_{\lambda\in \mathrm{sp}\,(T^l)} \lambda$ and we may assume that $\{\lambda_m(T^l)\} = \{\lambda_m(T)^l\}$, we get that trace $T^{kd+r} = 0$.

In proving inverse, we may (and do) assume that $Kd > N_0$. Consider an infinite increasing sequence $\{p_m\}$ of prime numbers with $p_1 > (K+1)d$. Assuming that trace $T^{kd+r} = 0$ if $k = K, K+1, K+2, \dots$ and $r = 1, 2, \dots, d-1$, for a fixed p_m we get from Lemma 3 (more precisely, from Corollary 4) that the function $\det(1 - zT^{p_m})$ is d-even. Suppose that the spectrum of T is not $\mathbb{Z}_d$-symmetric. Then there exist an eigenvalue $\lambda_0 \in \mathrm{sp}\,(T)$ and a root $\theta \in \sqrt[d]{1}$ so that $\theta\lambda_0 \notin \mathrm{sp}\,(T)$. On the other hand, again by Lemma 3, the spectrum of T^{p_m} is $\mathbb{Z}_d$-symmetric. Since $\lambda_0^{p_m} \in \mathrm{sp}\,(T^{p_m})$, there exists $\mu_m \in \mathrm{sp}\,(T)$ such that $\mu_m^{p_m} = \theta^{p_m}\lambda_0^{p_m} \in \mathrm{sp}\,(T^{p_m})$; hence, $\mu_m = \theta_m\lambda_0$ for some θ_m with $|\theta_m| = 1$. But $|\lambda_0| > 0$. Therefore, the set $\{\mu_m\}$ is finite and it follows that there is an integer $M > 1$ such that $\theta_M = \theta_{M+1} = \theta_{M+2} = \dots$. Hence, $\theta^{p_m} = \theta_M^{p_m}$ for all $m \geq M$. Thus, $\theta_M = \theta$. Contradiction.

Let us give some examples in which we can apply Theorem 4, but the main result of [6, 7] does not work.

Example 2. Let Π_p be the ideal of absolutely p-summing operators ($p \in [1, \infty)$; see [8] for a definition and related facts). Then for some n one has $\Pi_p^n \subset N$. In particular, $\Pi_2^2(C[0,1]) \subset N(C[0,1])$, but not every absolutely 2-summing operator in $C[0,1]$ is compact. Another interesting example: Π_3^3 is of spectral type l_1 [15]. We do not know (maybe, it is not known to everybody), whether the finite rank operators are dense in this ideal. However, Theorem 4 may be applied. Moreover, it can be seen that, for example, the spectrum of an operator T from Π_3^3 is central symmetric iff *the spectral traces* of the operators T^{2k-1} are zero for all $k > 0$.

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