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On zeroes and poles of Helson zeta functions

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ABSTRACT

We show that the analytic continuations of Helson zeta functions $\zeta_\chi(s) = \sum_1^\infty \chi(n)n^{-s}$ can have essentially arbitrary poles and zeroes in the strip $21/40 < \Re s < 1$ (unconditionally), and in the whole critical strip $1/2 < \Re s < 1$ under Riemann Hypothesis.

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Let $\chi\colon \mathbb{N} \to \mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ be a completely multiplicative function, and let

$$\zeta_\chi(s) = \sum_1^\infty \chi(n)n^{-s}. \tag{1}$$

This series defines an analytic function in the halfplane $\Re s > 1$. The Euler product formula,

$$\zeta_\chi(s) = \prod_p \frac{1}{1 - \chi(p)p^{-s}},$$

holds. In particular, the function ζ_χ has no zeroes for $\Re s > 1$.

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The study of functions ζ_χ was initiated by H. Helson who showed [3] in 1969 that for almost all χ's the function ζ_χ extends analytically to the halfplane $\Re s > 1/2$ and has no zeroes in this halfplane. Almost all here refers to the measure on the functions χ induced by the standard product measure on the infinite dimensional torus $\mathbb{T}^\infty$ via identification of χ with the sequence $\{\chi(p)\} \in \mathbb{T}^\infty$ of its values at primes. Recently it has been shown [4] that Helson's result is optimal in that the function ζ_χ does not admit meromorphic extension to the halfplane $\Re s > \alpha$ for any $\alpha < 1/2$ almost surely in the above sense.

These results bring about the question of the structure of analytic continuation for χ from the exceptional set in the Helson theorem. Such a study has been undertaken in [5] where it was shown that the set of zeroes of the meromorphic continuation of ζ_χ in the strip $1/2 < \Re s < 1$ can be essentially arbitrary. Namely, the following theorem holds.

Theorem 1. *[5, Theorem 1.4] For any set $\mathfrak{O}$ in the strip $1/2 < \Re s < 39/40$ which has no accumulation points off the line $\Re s = 1/2$ there exists a completely multiplicative function χ such that the Helson zeta function ζ_χ admits meromophic extension to the halfplane $\Re s > 1/2$, and*

$$\left\{ s : \zeta_\chi(s) = 0, \Re s > \frac{1}{2} \right\} = \mathfrak{O}.$$

If the RH is assumed then the same assertion holds with 1 in the place of 39/40.

This theorem is proved in [5] by constructing the required function χ. Given the zero set $\mathfrak{O}$, the construction in [5] involves forming "dipoles" made of a zero $\rho \in \mathfrak{O}$ and an additional pole nearby. The resulting function ζ_χ is meromorphic, and the set of its poles is not arbitrary. A natural further question mentioned in [5] is if there are any restrictions on the set of poles of ζ_χ in Theorem 1. Our result is the following theorem saying essentially that the set of poles can also be arbitrary.

To account for multiple zeroes and poles we are going to use the terminology of multisets. A multiset is a pair made of a subset $S \subset \mathbb{C}$ and a function $S \longrightarrow \mathbb{N}$. When we say that the set of zeroes of an analytic function f coincides with a multiset, M_S, we mean that $f(s) = 0$ iff $s \in S$, and the multiplicity of zero at $s \in S$ is $m(s)$. A similar terminology is used for poles.

Our goal is to establish the following

Theorem 2. *1. For any disjoint multisets Z and P in the strip $21/40 < \Re s < 1$ having no accumulation points off the line $\Re s = 21/40$ there exists a completely multiplicative function χ such that ζ_χ admits meromorphic continuation to the halfplane $\Re s > 21/40$ with Z the set of its zeroes and P the set of its poles.*

2. If RH holds then for any disjoint multisets Z and P in the strip $1/2 < \Re s < 1$ having no accumulation points off the line $\Re s = 1/2$ there exists a completely multiplicative function χ such that ζ_χ admits meromorphic continuation to the halfplane $\Re s > 1/2$ with Z the set of its zeroes and P the set of its poles.

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Logarithmic derivative

The strategy of the proof is as follows. Let ζ_χ be the function we seek to find. Consider $g(s) = \frac{\zeta'_\chi(s)}{\zeta_\chi(s)}$. From the Euler product representation we have

$$g(s) = -\sum_{n=1}^{\infty} \chi(n)\Lambda(n)n^{-s} = -\sum_{p,a} \chi(p^a)\Lambda(p^a)p^{-as} = \\ -\sum_{p} \chi(p)\Lambda(p)p^{-s} - \sum_{p,a\geq 2} \chi(p^a)\Lambda(p^a)p^{-as},$$

where p ranges over the primes, a over the naturals, and Λ is the von Mangoldt function. The second sum in the right hand side is absolutely convergent for $\Re s > 1/2$, hence the poles and residues of g are exactly those of the function

$$\tilde{g}(s) = -\sum_{p} \chi(p)p^{-s}\log p.$$

Thus the problem is reduced to the one of constructing the function $\tilde{g}$ with the required poles and residues.

Mellin transform

We seek the function $\tilde{g}$ in the form

$$\tilde{g}(s) = h(s) + \int_1^{\infty} q(x)x^{-s}dx, \Re s > 1,$$

where h is analytic in the halfplane $\Re s > 1/2$, and $q(x) = o(1)$, $x \to +\infty$.

The following lemma extracts from [5, 8.1] a part of the argument we are going to need. A proof is provided for completeness.

Lemma 1. *Let $q(x) = o(1)$, $x \to +\infty$. Then there exists a completely multiplicative function χ such that the function*

$$\int_1^{\infty} q(x)x^{-s}dx - \sum_{p} \chi(p)p^{-s}\log p,$$

initially defined in the halfplane $\Re s > 1$, extends analytically to the halfplane $\Re s > 21/40$ unconditionally, and to the halfplane $\Re s > 1/2$ if the RH holds.

Proof. Arguing as in [5, 8.1] we are going to construct χ for a subset, $\mathcal{P}$, of primes using the identity

$$\int_1^\infty q(x)x^{-s}dx - \sum_{p\in\mathcal{P}} \chi(p)p^{-s}\log p =$$

$$s\int_1^\infty \left(\int_1^x q(y)dy - \sum_{p\le x, p\in\mathcal{P}} \chi(p)\log p\right) x^{-s-1}dx.$$

It suffices to show that there exists a χ such that

$$r(x) := \int_1^x q(y)dy - \sum_{p\le x, p\in\mathcal{P}} \chi(p)\log p = O\big(x^{\frac{21}{40}}\log x\big). \tag{2}$$

Let

$$x_0 = 2, x_{j+1} = x_j + x_j^{\frac{21}{40}}. \tag{3}$$

It is enough then to establish (2) at the sequence $x = x_j$. Indeed, if it is satisfied for $x = x_j$ then for $x \in [x_j, x_{j+1})$ we have

$$r(x) = r(x_j) + \int_{x_j}^x q(y)dy - \sum_{x_j<p\le x, p\in\mathcal{P}} \chi(p)\log p.$$

The first term in rhs is $O(x^{21/40}\log x)$ by assumption,

$$\int_{x_j}^x q(y)dy = O(x^{\frac{21}{40}})$$

by the boundedness of q, and the rightmost term in the right hand side is trivially $O(x^{21/40}\log x)$.

It remains to choose χ so that $r(x_j) = O\big(x_j^{21/40}\log x_j\big)$. In fact we are going to choose it so that $r(x_j) = O(\log x_j)$. This is done by induction in j. Let a subset $\mathcal{P}\cap[0, x_j)$ and the function χ on it be constructed so that $r(x_k) \le C\log x_k$ for some C for $k = 1, 2, \dots, j$. Clearly

$$\int_{x_j}^{x_{j+1}} q(y)dy = o\big(x_j^{\frac{21}{40}}\big). \tag{4}$$

Let c_j be the argument of the number $r(x_j) + \int_{x_j}^{x_{j+1}} q$, and let $\chi(p) = e^{ic_j}$, $p \in \mathcal{P}_j$, where $\mathcal{P}_j$ is a subset of primes on the interval $[x_j, x_{j+1})$ to be chosen as follows. By construction we have

$$|r(x_{j+1})| = \left| \left| r(x_j) + \int_{x_j}^{x_{j+1}} q \right| - \sum_{p \in \mathcal{P}_j} \log p \right|.$$

Starting from an empty set, we shall add numbers to the $\mathcal{P}_j$ until the right hand side becomes $\le \log x_{j+1}$. This is possible since

$$\left| r(x_j) + \int_{x_j}^{x_{j+1}} q \right| = o\big(x_j^{\frac{21}{40}}\big)$$

by the induction assumption and (4), while the interval $[x_j, x_{j+1})$ contains at least $C_1 x_j^{21/40} / \log x_j$ primes ([1], p. 562) for some constant $C_1 > 0$ independent of j. Thus $r_{j+1} \le C \log x_{j+1}$, as required.

The construction above defines χ on a subset $\mathcal{P} = \cup_j \mathcal{P}_j$ of primes. Extending it to all primes is done verbatim as in [5, Lemma 5.3]. This proves the unconditional part of the assertion.

Assuming the Riemann hypothesis we follow the same argument with $x_{i+1} = x_i + 4x_i^{1/2} \log x_i$ instead of (3), and take into account that the number of primes in the interval $[x, x + c\sqrt{x} \log x]$ is estimated below by $\sqrt{x}$ for all $c > 3$ [2]. This gives

$$r(x) = O(\sqrt{x} \log^2 x)$$

which implies the required assertion in the conditional case. □

Thus it remains to find an analytic function g_1 in the halfplane $\Re s > 1$ of the form

$$g_1(s) = \int_1^\infty q(x) x^{-s} dx,$$

with q vanishing as $x \to +\infty$ which admits meromorphic extension to the halfplane $\Re s > 21/40$ with the prescribed poles and respective residues.

Lemma 2. *Let g be an analytic function in the halfplane $\Re z > 1$ such that $\sup |z|^2 |g(z)| < \infty$. Then there exists a continuous function, q, $q(x) = o(1)$, $x \to +\infty$, such that*

$$g(s) = \int_1^\infty q(x) x^{-s} dx, \ \Re s > 1.$$

Proof. Consider the function $h(t) = g(-it+1)$. The function h belongs to the Hardy class H^2_+, hence the restriction $h|_{\mathbb{R}}$ is the inverse Fourier transform of a certain function $p \in L^2(\mathbb{R})$, vanishing on the negative real axis. Since $h|_{\mathbb{R}} \in L^2 \cap L^1$, the function p is the classical Fourier transform of h, hence it is continuous and vanishes as $x \to +\infty$ by the Riemann–Lebesgue lemma. The function $q(s) := p(\log s)$ then also vanishes as $s \to +\infty$. Finally we have (recall that $s = -it+1$):

$$\int_1^\infty q(x)x^{-s}dx = \int_0^\infty q(e^y)e^{(1-s)y}dy = \int_0^\infty p(y)e^{ity}dy = h(t) = g(s),$$

as required. □

Mellin transforms with prescribed poles and residues in a strip

In this section we construct the function g. Let $\alpha = 21/40$ in the unconditional case, and $\alpha = 1/2$ if RH is satisfied. Assume first that the sets Z and P have no accumulation points at finite distance. In view of Lemma 2, Theorem 2 for this case will be proven if we manage to find a function g analytic in the halfplane $\Re s > 1$ and satisfying $\sup_{\Re z>1} |g(z)|\,|z|^2 < \infty$, which admits a meromorphic extension to the halfplane $\Re z > \alpha$ with the given poles and residues in the strip $\alpha < \Re z < 1$.

Notice first that, given a point z_0, $\alpha < \Re z_0 < 1$, and a number $C > 0$, one can choose $n = n(C, z_0)$ large enough so that the function

$$g_{z_0}(z) = \frac{1}{(z-z_0)(z-z_0+1)^{2n}}$$

has the following properties,

(i) $|g_{z_0}(z)| < C$ for $\Re z > 1$.

(ii) g_{z_0} is analytic in $\{\Re z > \alpha\}$ except at z_0, has a simple pole at z_0 with $\operatorname{Res}_{z_0} g_{z_0} = 1$.

(iii) $|g_{z_0}(z)| < C$ for $|z - z_0| > 3$, $\Re z > \alpha$.

Lemma 3. *Let Σ be a subset of the strip $\alpha < \Re z < 1$ having no accumulation points at finite distance, and $m\colon \Sigma \longrightarrow \mathbb{C} \setminus \{0\}$ be an arbitrary function. Then there exists a meromorphic function g in the halfplane $\Re z > \alpha$, whose set of poles coincides with Σ, all poles are simple, $\operatorname{Res}_z g = m(z)$ for $z \in \Sigma$, and*

$$\sup_{\Re z>1} |g(z)||z|^2 < \infty.$$

Note that the function m needed for Theorem 2 is in fact integer valued.

Proof. Let G_1 be an arbitrary function analytic in the halfplane $\Re z > \alpha$, having no zeroes and satisfying $G_1(z) = O(|z|^{-2})$ as $|z| \to \infty$. $G_1(z) = e^{-z}z^{-2}$ will do.

Fix an arbitrary enumeration of Σ. Given a $p_i \in \Sigma$, define g_i to be the function g_{p_i} satisfying properties (i)–(iii) with

$$C = \frac{|G_1(p_i)|}{|m(p_i)|2^{i+1}}.$$

Let

$$g(z) = G_1(z) \sum_i m(p_i) \frac{g_i(z)}{G_1(p_i)}. \tag{5}$$

Let us first check that the series in the right hand side converge absolutely at any point $z \notin \Sigma$ in the halfplane $\Re z > \alpha$, the convergence being uniform on compacts in $\{\Re z > \alpha\} \setminus \Sigma$, and thus the function g is meromorphic with simple poles at Σ and no other singularities.

Indeed let $z \notin \Sigma$, $\Re z > \alpha$. Clearly $|z - p_i| \le 3$ for at most finitely many i's. If $|z - p_k| > 3$ for some k, then

$$|g_k(z)| < \frac{|G_1(p_k)|}{|m(p_k)|2^{k+1}},$$

by (iii) from whence

$$\left| m(p_k) \frac{g_k(z)}{G_1(p_k)} \right| < 2^{-k-1},$$

and the convergence is proven. The equality $\operatorname{Res}_{p_i} g = m(p_i)$ is obvious. It remains to notice that for $\Re z > 1$ we have

$$\left| m(p_i) \frac{g_i(z)}{G_1(p_i)} \right| \le 2^{-i-1},$$

hence $|g(z)| \le |G_1(z)|$, and thus $\sup_{\Re z \ge 1} |z|^2 |g(z)| < \infty$, as required. □

Theorem 2 is thus proved in the partial case when the sets Z and P do not accumulate at finite distance. The general case is reduced to this one via dyadic decomposition of the strip in the same way as in [5, Section 8.1].

As a final remark, let us comment on the shift of the strip by 1/40 in Theorem 2 to the right as compared to the original result of Seip (Theorem 1). The construction used in [5] actually gives a meromorphic continuation to the strip of width $\Delta = 1 - \sigma_*$ where σ_* is the best order in the theorem on primes in intervals (currently $\sigma_* = 21/40$) in the sense that if the set Z (subject to restrictions analogous to those in Theorem 2) lies in the strip $\max\{\sigma_1 - \Delta, 1/2\} < \Re s < \sigma_1$, $\sigma_1 \le 1$, then it allows to construct a ζ_χ analytic in $\Re s > \sigma_1$ and meromorphic in the halfplane $\Re s > \max\{\sigma_1 - \Delta, 1/2\}$ with Z being the set of its zeroes. For $\sigma_1 = 1$ this would proof the part of Theorem 2 related

to zeroes. The actual assertion in [5, Theorem 1.4] sets $\sigma_1 = 39/40$ and this gives the shift. The construction in [5] "reverse engineers" the function q of Lemma 1 above from its explicitly given Mellin transform, R, so that h has power decay of order $1 - \sigma_1$, $q(x) = O(x^{-1/40} \log^2 x)$ for the chosen σ_1, and it is this decay that leads to the left bound being $\sigma_1 - \Delta$. Our construction is done in terms of q rather than R. The price to pay is that we got no control over the decay of q in the power scale.

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