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RESEARCH ARTICLE

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Helson zeta functions for characters with finitely many values

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Abstract

We show that the analytic continuations of Helson zeta functions $\zeta_\chi(s) = \sum_1^\infty \chi(n)n^{-s}$ can have essentially arbitrary poles and zeroes in the strip $21/40 < \Re s < 1$ (unconditionally), and in the whole critical strip $1/2 < \Re s < 1$ under Riemann Hypothesis, with the function χ taking values in cubic roots of unity. If the sets of zeroes and poles are symmetric with respect to the real axis, the same can be achieved with χ taking values ± 1. The proof is constructive.

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Let $\chi : \mathbb{N} \to \mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, be a totally multiplicative function. The Helson zeta function ζ_χ is defined as follows:

$$\zeta_\chi(s) = \sum_1^\infty \chi(n)n^{-s}. \tag{1}$$

With this definition, ζ_χ is an analytic function in the halfplane $\Re s > 1$. It satisfies the Euler product formula,

$$\zeta_\chi(s) = \prod_p \frac{1}{1 - \chi(p)p^{-s}}.$$

In particular, the function ζ_χ has no zeroes for $\Re s > 1$.

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The interest to this particular generalization of the Riemann zeta function stems from a result of H. Helson [4], asserting that for almost all χ's, the function ζ_χ extends analytically to the halfplane $\Re s > 1/2$ and the extension has no zeroes in this halfplane. Here "almost all" refers to the measure on the functions χ induced by the standard product measure on the infinite dimensional torus $\mathbb{T}^\infty$ via identification of χ with the sequence $\{\chi(p)\} \in \mathbb{T}^\infty$ of its values at primes.

In the sense of Helson's theorem, the Riemann zeta function is in the exceptional set because of its pole at 0. This calls for further study of analytic continuation of Helson zeta function for χ's from the exceptional set. The result of this paper is the following theorem, showing that the meromorphic extension of χ can have essentially arbitrary zeroes and poles in the critical strip.

Given a set P and a function $m_P : P \to \mathbb{N}$, let $\mathcal{P}(P, m_P)$ be the set of meromorphic functions in a halfplane with poles in P and such that the multiplicity of the pole of a function from $\mathcal{P}(P, m_P)$ at $z \in P$ is $m_P(z)$. The set $\mathcal{Z}(Z, m_Z)$ is defined in a similar way, wth zeroes instead of poles.

Theorem 1. *Let Z and P be arbitrary disjoint sets in the strip $21/40 < \Re s < 1$ having no accumulation points off the line $\Re s = 21/40$, and let $m_Z : Z \to \mathbb{N}$ and $m_P : P \to \mathbb{N}$ be arbitrary functions.*

1. *There exists a totally multiplicative function $\chi : \mathbb{N} \longrightarrow \{e^{\pm 2\pi i/3}, 1\}$ such that ζ_χ admits meromorphic continuation to the halfplane $\Re s > 21/40$ belonging to $\mathcal{P}(P, m_P) \cap \mathcal{Z}(Z, m_Z)$.*
2. *If additionally the sets Z and P and functions m_Z and m_P are symmetric with respect to the real line, then there exists a totally multiplicative function $\chi : \mathbb{N} \longrightarrow \{\pm 1\}$ such that ζ_χ admits meromorphic continuation to the halfplane $\Re s > 21/40$ belonging to $\mathcal{P}(P, m_P) \cap \mathcal{Z}(Z, m_Z)$.*
3. *If the Riemann hypothesis holds, then both assertions 1 and 2 stay true with $21/40$ replaced by $1/2$.*

This theorem is optimal in several senses.

- The symmetricity assumption is obviously necessary in the case of χ taking values ± 1, for the function ζ_χ is real on the real line in this situation. This means that the result is nonimprovable as far as the range of χ is concerned.
- The proof is constructive — we give a recurrent algorithm of assigning one of the three (or two, depending on the case) values to $\chi(p)$ for each prime p.

Notice also that the first part of Theorem 1 holds, with the same proof, if the set $\{e^{\pm 2\pi i/3}, 1\}$ in the formulation is replaced with $\{z : z^l = 1\}$, for any $l > 3$.

The modern history of the problem starts from K. Seip's paper [6] where a result of this type has been established for zeroes of the meromorphic continuation.

Theorem 2 [6, Theorem 1.4]. *For any set $\mathfrak{O}$ in the strip $1/2 < \Re s < 39/40$ which has no accumulation points off the line $\Re s = 1/2$, there exists a totally multiplicative function χ such that the Helson zeta function ζ_χ admits meromophic extension to the halfplane $\Re s > 1/2$, and*

$$\left\{ s : \zeta_\chi(s) = 0, \Re s > \frac{1}{2} \right\} = \mathfrak{O}.$$

If the Riemann Hypothesis is assumed, then the same assertion holds with 1 in the place of $39/40$.

The proof of this theorem in [6] gives ζ_χ in the form of a product of two functions, $\zeta_\chi = r_1 r_2$. The function r_1 is determined by the values of χ on a subset, $\mathcal{P}$, of primes. The set $\mathcal{P}$ and the restriction $\chi|_{\mathcal{P}}$ are defined explicitly in such a way that r_1 has the required analytic extension with the zero set $\mathfrak{D}$, while r_2 is determined by the values of χ on the rest of the primes and has zero-free analytic extension up to the critical line $\Re s = 1/2$. Notice that the function r_2 is constructed by probabilistic methods of the Helson's theorem rendering the overall assertion nonconstructive. The construction of r_1 uses "dipoles" of the form $(z-\rho)^{-1} - (z-\rho')^{-1}$, where $\rho \in \mathfrak{D}$ and ρ' is a pole nearby ρ, as building blocks for its logarithmic derivative. The set of poles occurring in this way is not controlled.

A way to tackle zeroes and poles simultaneously was found in our previous paper [2] where a weaker version of Theorem 1 has been established. In that work, the existence of characters $\chi : \mathbb{N} \to \mathbb{T}$ with the given sets of zeroes and poles was established for the same strips as in Theorem 1. The argument, however, did not allow to control the range of χ. Also, the proof in [2] still relied on the multiplicative representation $\zeta_\chi = r_1 r_2$ of Seip's work [6] with r_2 defined by probabilistic arguments, and thus, was nonconstructive.

The proof of Theorem 1 is based on (i) a novel construction of the character χ that allows to approximate the Mellin transform of a function using the logarithmic derivative of ζ_χ, and (ii) a special interpolation procedure combined with the elementary Hardy classes theory allowing to control poles.

Finally, let us notice that an analog of the Helson theorem can be shown to hold for the measure generated by purely discrete measure on $\mathbb{T}$ supported on roots of unity and giving the equal weight to every root. The characters χ constructed in Theorem 1 thus lie in the exceptional set even with respect to this finer measure.

Throughout the paper, all the sums with summation variable p range over primes in an interval specified in the sum limits. No limits means that the sum ranges over all primes.

1 | LOGARITHMIC DERIVATIVE

Let ζ_χ be a Helson zeta function. From the Euler product representation, we have

$$\begin{aligned}\frac{\zeta_\chi'(s)}{\zeta_\chi(s)} &= -\sum_{n=1}^{\infty} \chi(n)\Lambda(n)n^{-s} = -\sum_{p,a} \chi(p^a)\Lambda(p^a)p^{-as} \\ &= -\sum_{p} \chi(p)\Lambda(p)p^{-s} - \sum_{p,a\geqslant 2} \chi(p^a)\Lambda(p^a)p^{-as},\end{aligned}$$

where p ranges over the primes, a over the naturals, and Λ is the von Mangoldt function. The second sum in the right-hand side is absolutely convergent for $\Re s > 1/2$; hence, the left-hand side admits meromophic extension to the halfplane $\Re s > \alpha$, $\alpha \geqslant 1/2$, if and only if the function

$$\tilde{g}(s) = -\sum_{p} \chi(p)p^{-s}\log p \tag{2}$$

does. These extensions then have the same poles and residues.

Thus, the problem is reduced to the one of constructing the function $\tilde{g}$ with the required poles and residues. We are going to seek the function $\tilde{g}$ in the form

$$\tilde{g}(s) = h(s) + \int_1^\infty q(x)x^{-s}dx, \Re s > 1,$$

where h is analytic in the halfplane $\Re s > 1/2$, and $q(x) = o(1)$, $x \to +\infty$.

Lemma 1. *Let q be a continuous function on $[1, +\infty)$, and $q(x) = o(1)$, $x \to +\infty$.*

Then there exists a totally multiplicative function $\chi : \mathbb{N} \longrightarrow \{e^{\pm 2\pi i/3}, 1\}$ such that the function

$$\int_1^\infty q(x)x^{-s}dx - \sum_p \chi(p)p^{-s}\log p,$$

initially defined in the halfplane $\Re s > 1$, extends analytically to the halfplane $\Re s > 21/40$ unconditionally, and to the halfplane $\Re s > 1/2$ if the RH holds.

If additionally, the function q is real, then there exists a totally multiplicative function $\chi : \mathbb{N} \longrightarrow \{\pm 1\}$ with the same properties.

Proof. Arguing as in [6, 8.1], we use the identity

$$\int_1^\infty q(x)x^{-s}dx - \sum_p \chi(p)p^{-s}\log p$$
$$= s\int_1^\infty \left(\int_1^x q(y)dy - \sum_{p\leqslant x}\chi(p)\log p\right)x^{-s-1}dx.$$

It suffices to show that there exists a χ such that

$$r(x) := \int_1^x q(y)dy - \sum_{p\leqslant x}\chi(p)\log p = O\left(x^{\frac{21}{40}}\log x\right). \tag{3}$$

Let

$$x_0 = 2, x_{j+1} = x_j + x_j^{\frac{21}{40}}. \tag{4}$$

It is enough then to establish (3) at the sequence $x = x_j$. Indeed, if it is satisfied for $x = x_j$, then for $x \in [x_j, x_{j+1})$, we have

$$r(x) = r(x_j) + \int_{x_j}^x q(y)dy - \sum_{x_j<p\leqslant x}\chi(p)\log p.$$

The first term in the right-hand side is $O(x^{21/40}\log x)$ by assumption,

$$\int_{x_j}^x q(y)dy = O\left(x^{\frac{21}{40}}\right)$$

by the boundedness of q, and the rightmost term in the right-hand side is trivially $O(x^{21/40}\log x)$.

It remains to choose χ so that $r(x_j) = O\big(x_j^{21/40}\log x_j\big)$. In fact, we are going to choose it so that $r(x_j) = O(\log x_j)$. This is done by induction in j.

We are going to choose $\chi(p)$ for primes $p \in [x_j, x_{j+1})$ so that

$$|r(x_{j+1})| \leqslant \max\big\{|r(x_j)|, 3\log x_{j+1}\big\}$$

for all j large enough.

Denote by $c_1 \in [-\pi, \pi)$ the argument of the number

$$\rho_j := r(x_j) + \int_{x_j}^{x_{j+1}} q,$$

and let $k_1 \in \{0, \pm 1\}$ be chosen so that $|2k_1\pi/3 - c_1| \leqslant \pi/3$. Let p_1 be the smallest prime in $[x_j, x_{j+1})$, and define $\chi(p_1) = e^{2ik_1\pi/3}$. Then we take the number $\rho_j - \chi(p_1)\log p_1$, define $c_2 \in [-\pi, \pi)$ to be its argument, choose $k_2 \in \{0, \pm 1\}$ so that $|2k_2\pi/3 - c_2| \leqslant \pi/3$, and let $\chi(p_2) = e^{2ik_2\pi/3}$ with p_2 the smallest prime in $[x_j, x_{j+1})$ larger than p_1, and so on, through all primes in $[x_j, x_{j+1})$. We have

$$\begin{aligned} r(x_{j+1}) &= r(x_j) + \int_{x_j}^{x_{j+1}} q - \sum_{p\in[x_j, x_{j+1})} \chi(p)\log p \\ &= \rho_j - e^{2ik_1\pi/3}\log p_1 - \sum_{p\in(p_1, x_{j+1})} \chi(p)\log p. \end{aligned}$$

Then, by elementary trigonometry,

$$\begin{aligned} \Big|\rho_j - e^{2ik_1\pi/3}\log p_1\Big| &= \Big||\rho_j| - e^{i(2k_1\pi/3 - c_1)}\log p_1\Big| \\ &\leqslant \begin{cases} |\rho_j| - (\log p_1)/4, & \log p_1 \leqslant |\rho_j|/2, \\ 3\log p_1, & \log p_1 > |\rho_j|/2, \end{cases} \end{aligned} \tag{5}$$

because $2\cos(2k_1\pi/3 - c_1) \geqslant 1$, and $|a - e^{i\pi/3}b| \leqslant a - b/4$ for real a, b whenever $0 \leqslant b \leqslant a/2$. Repeating this estimate with $\rho_j^1 = \rho_j - e^{2ik_1\pi/3}\log p_1$ in the place of ρ_j, we get

$$\Big|\rho_j^1 - e^{2ik_2\pi/3}\log p_2\Big| \leqslant \max\left\{\Big|\rho_j\Big| - \frac{\log p_1 + \log p_2}{4}, 3\log p_2\right\},$$

and so on. We are now going to use the fact that the interval $[x_j, x_{j+1})$ contains at least $C_1 x_j^{21/40}/\log x_j$ primes ([1], p. 562) for some constant $C_1 > 0$ independent of j. It shows that eventually, after all primes on the interval $[x_j, x_{j+1})$ are accounted for, the first of the two numbers over which the maximum is taken, $|\rho_j| - (\log p_1 + \log p_2 + \ldots)/4$, is not greater than $|\rho_j| - C_1 x_j^{21/40}/4$.

The other number in the maximum is obviously estimated above by $3\log p_* \leqslant 3\log x_{j+1}$, p_* being the largest prime on $[x_j, x_{j+1})$. Thus,

$$|r(x_{j+1})| \leqslant \max\Big\{|\rho_j| - C_1 x_j^{21/40}/4, 3\log x_{j+1}\Big\}.$$

On the other hand,

$$|\rho_j| \leqslant |r(x_j)| + o\left(x_j^{21/40}\right)$$

since q is assumed to vanish at infinity. Picking M large enough so that the $o(x^{21/40})$-term gets smaller than $C_1 x_i^{21/40}/8$ for $i \leqslant M$, we obtain that whenever $j \geqslant M$

$$|r(x_{j+1})| \leqslant \max\left\{|r(x_j)|, 3\log x_{j+1}\right\},$$

as required. Thus, $|r(x_{j+1})| = O(\log x_{j+1})$. For the implied constant, one can take the maximum over $|r(x_i)|/\log x_i$, $i \leqslant M$, and 3. This proves the unconditional part of the assertion.

Assuming the Riemann hypothesis, we follow the same argument with $x_{i+1} = x_i + 4x_i^{1/2}\log x_i$ instead of (4), and take into account that the number of primes in the interval $[x, x + c\sqrt{x}\log x]$ is estimated below by $\sqrt{x}$ for all $c > 3$ [3]. This gives

$$r(x) = O(\sqrt{x}\log^2 x),$$

which implies the required assertion in the conditional case.

The part of the assertion pertaining to the case of real q is proved similarly but easier — in this case instead of (5), we are going to have simply

$$\left|\rho_j - \chi(p_1)\log p_1\right| = ||\rho_j| - \log p_1||. \qquad \square$$

Thus, it remains to find an analytic function g_1 in the halfplane $\Re s > 1$ of the form

$$g_1(s) = \int_1^\infty q(x)x^{-s}dx,$$

with q vanishing as $x \to +\infty$, which admits meromorphic extension to the halfplane $\Re s > 21/40$ with the prescribed poles and respective residues.

Lemma 2. *Let g be an analytic function in the halfplane $\Re z > 1$ such that $\sup |z|^2|g(z)| < \infty$. Then there exists a continuous function, q, $q(s) = o(1)$, $s \to +\infty$, such that*

$$g(s) = \int_1^\infty q(s)x^{-s}dx, \ \Re s > 1.$$

If additionally $g(s)$ is real for real s, then q is real.

This lemma is essentially contained in [2]. The proof is reproduced here for completeness.

Proof. Consider the function $h(t) = g(-it + 1)$. The function h belongs to the Hardy class H_+^2 in the upper half plane; hence, the restriction $h|_{\mathbb{R}}$ is the inverse Fourier transform of a certain function $p \in L^2(\mathbb{R})$, vanishing on the negative real axis. Since $h|_{\mathbb{R}} \in L^2 \cap L^1$, the function p is the classical Fourier transform of h; hence, it is continuous and vanishes as $x \to +\infty$ by the Riemann–Lebesgue lemma. The function $q(s) := p(\log s)$ then also vanishes as $s \to +\infty$. Finally, we have

(recall that $s = -it + 1$):

$$\int_1^\infty q(x)x^{-s}dx = \int_0^\infty q(e^y)e^{(1-s)y}dy = \int_0^\infty p(y)e^{ity}dy = h(t) = g(s),$$

as required. It remains to notice that if $g(s)$ is real for real s, then $\overline{h(t)} = h(-t)$ and therefore p and q are real. □

2 | MELLIN TRANSFORMS WITH PRESCRIBED POLES AND RESIDUES IN A STRIP

Let $\alpha = 21/40$ in the unconditional case, and $\alpha = 1/2$ if RH is satisfied. Assume first that the sets Z and P have no finite accumulation points. In view of lemma 2, Theorem 1 for this case will be proven if we manage to find a function g analytic in the halfplane $\Re s > 1$ and satisfying $\sup_{\Re z>1} |g(z)|\,|z|^2 < \infty$, which admits meromorphic extension to the halfplane $\Re z > \alpha$ with the given poles and residues in the strip $\alpha < \Re z < 1$.

Notice first that, given a point z_0, $\alpha < \Re z_0 < 1$, and a number $C > 0$, one can choose $n = n(C, z_0)$ large enough so that the function

$$g_{z_0}(z) = \frac{1}{(z - z_0)(z - z_0 + 1)^{2n}}$$

has the following properties:

(i) $|g_{z_0}(z)| < C$ for $\Re z > 1$.
(ii) g_{z_0} is analytic in $\{\Re z > \alpha\}$ except at z_0, has a simple pole at z_0 with $\mathrm{Res}_{z_0}\, g_{z_0} = 1$.
(iii) $|g_{z_0}(z)| < C$ for $|z - z_0| > 3$, $\Re z > \alpha$.

Lemma 3. *Let Σ be a subset of the strip $\alpha < \Re z < 1$ having no finite accumulation points, and let $m : \Sigma \longrightarrow \mathbb{C} \setminus \{0\}$ be an arbitrary function. Then there exists a meromorphic function g in the halfplane $\Re z > \alpha$, whose set of poles coincides with Σ, all poles are simple, $\mathrm{Res}_z\, g = m(z)$ for $z \in \Sigma$, and*

$$\sup_{\Re z>1} |g(z)||z|^2 < \infty.$$

If additionally, the set Σ is symmetric with respect to the real line, m is real-valued, and $m(z) = m(\overline{z})$, then $g(z)$ is real for real z.

Proof. Let G_1 be an arbitrary zero-free function analytic in the halfplane $\Re z > \alpha$, real at real z, and satisfying $G_1(z) = O(|z|^{-2})$ as $|z| \to \infty$. $G_1(z) = e^{-z}z^{-2}$ will do.

Fix an arbitrary enumeration of Σ. Given a $p_i \in \Sigma$, define g_i to be the function g_{p_i} satisfying properties (i)–(iii) with

$$C = \frac{|G_1(p_i)|}{|m(p_i)|2^{i+1}}.$$

Let

$$g(z) = G_1(z) \sum_i m(p_i) \frac{g_i(z)}{G_1(p_i)}. \tag{6}$$

Let us first check that the series in the right-hand side converge absolutely at any point $z \notin \Sigma$ in the halfplane $\Re z > \alpha$, the convergence being uniform on compacts in $\{\Re z > \alpha\} \setminus \Sigma$, and thus, the function g is meromorphic with simple poles at Σ and no other singularities.

Indeed, let $z \notin \Sigma$, $\Re z > \alpha$. Clearly, $|z - p_i| \leqslant 3$ for at most finitely many i's. If $|z - p_k| > 3$ for some k, then

$$|g_k(z)| < \frac{|G_1(p_k)|}{|m(p_k)|2^{k+1}},$$

by (iii) from whence

$$\left| m(p_k) \frac{g_k(z)}{G_1(p_k)} \right| < 2^{-k-1},$$

and the convergence is proven. The equality $\operatorname{Res}_{p_i} g = m(p_i)$ is obvious. It remains to notice that for $\Re z > 1$, we have

$$\left| m(p_i) \frac{g_i(z)}{G_1(p_i)} \right| \leqslant 2^{-i-1},$$

hence $|g(z)| \leqslant |G_1(z)|$, and thus, $\sup_{\Re z \geqslant 1} |z|^2 |g(z)| < \infty$, as required.

The assertion about $g(z)$ being real for real z for symmetric Σ is immediate from (6). □

Theorem 1 is thus proved in the special case when the sets Z and P do not accumulate at finite distance. The general case is reduced to this one via dyadic decomposition of the strip in the same way as in [6, Section 8.1].

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