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Polynomial Birth–Death Processes and the 2nd Conjecture of Valent

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The conjecture of Valent about the type of Jacobi matrices with polynomially growing weights is proved.

1 Introduction

A moment problem, associated to a birth–death process with birth rates λ_n, $n \geq 0$, and death rates μ_n, is a symmetric Jacobi matrix of the form ($\lambda_n > 0$ for $n \geq 0$, $\mu_n > 0$ for $n > 0$, $\mu_0 \geq 0$)

$$J = \begin{pmatrix} a_0 & b_0 & 0 & \cdots & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ \cdots & 0 & b_2 & a_3 & \cdots \\ \cdots & \cdots & \cdot & \cdots & \cdots \end{pmatrix},$$

with $a_n = \lambda_n + \mu_n$, $b_n = \sqrt{\lambda_n \mu_{n+1}}$ [1]. The symmetric operator defined by the matrix J in the Hilbert space l^2 is either essentially self-adjoint or has the deficiency indices $(1, 1)$ [7]. In the latter case, the matrix J is said to be limit-circle (or Hamburger indeterminate). If J is limit-circle then its self-adjoint extensions have discrete spectra,

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and the spectra of different extensions are interlacing. Whenever the matrix J is limit-circle, the order and the corresponding type of a process with rates λ_n, μ_n are defined as the convergence exponent of the sequence $|t_n|$ where t_n ranges over the spectrum of a self-adjoint extension of J and the type of the corresponding Weierstraß product, respectively. Because of the interlacing property, the order and the type are independent of the choice of the extension, rendering the definition correct. Moreover, one can use it to define the order and the type for any limit-circle Jacobi matrix. According to a theorem of M. Riesz [7], the Weierstraß product is of minimal type with respect to order 1, hence the order is ≤ 1.

Three explicit examples of Jacobi matrices of orders in between 0 and 1, for which the types with respect to their orders are known, have been found so far [11–13]. Two of them [11, 12] correspond to some birth–death processes with quartic and cubic polynomial rates, and all of them are explicitly solvable in a sense. On the basis of the 1st two examples, in 1998, G. Valent in [2] conjectured the order and type of birth and death processes having polynomial birth and death rates of degree $p \geq 3$. His conjecture says that the order of the birth–death processes with rates λ_n, μ_n being the polynomials of degree p,

$$\lambda_n = n^p + Cn^{p-1} + \dots,$$

$$\mu_n = n^p + Dn^{p-1} + \dots,$$

subject to the condition $1 < C - D < p - 1$ is $1/p$, and its type, t_p, with respect to that order is

$$p \int_0^1 (1 - x^p)^{-2/p} \mathrm{d}x. \tag{1}$$

This form of the hypothesis in [2] refers to the case $\mu_0 = 0$, but since the order and type do not depend on compact perturbations of J_p, one can omit this assumption. The assumption about C and D ensures that the matrix J_p is limit-circle.

The analysis of the two mentioned examples [11, 12] is based on explicit solutions for the corresponding spectral problems and its methods are most likely not suitable to establish the Valent conjecture in full generality. Tellingly, in these examples, the rates λ_n and μ_n are polynomials of degrees $p = 3$ and $p = 4$ with specific numerical values of coefficients, and there are no free parameters.

In [6], the conjecture has been reduced to the following question. Let $a_j = j^p$, $p > 1, j \geq 1$, and let

$$L_p = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ \cdots & 0 & a_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

This matrix is well-known to be limit-circle [7, Chapter 1, Exercise 5]. As shown in [6], the Valent conjecture holds true if the order of L_p is $\frac{1}{p}$ and the type with respect to that order is $\frac{1}{2}B(1/(2p), 1 - 1/p)$. Here and throughout the paper B is the Euler beta-function, $B(x, y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t$.

The conjecture about the order was proved by Romanov in [4] as an application of a general method of estimating the order of canonical systems developed in that paper. In [6], Berg and Szwarc gave another proof of the order conjecture and established that the type satisfies

$$\frac{\pi}{\sin(\pi/p)} \leq t_p \leq \frac{\pi}{\sin(\pi/p)\cos(\pi/p)}.$$

This estimate is compatible with the conjecture about type in the sense that the quantity (1) satisfies this inequality.

In the present paper, the 2nd Valent conjecture is proved completely. Our proof uses an expression for the type due to Berg and Szwarc (a minor misprint in the formulation is corrected).

Theorem 1 ([6], Theorem 1.11).

$$t_p = \frac{p}{e} \limsup_{n\to\infty} \left(n \left(\sum_{1 \leq x_1 \leq x_2 < x_3 \leq x_4 < \ldots < x_{2n-1} \leq x_{2n}} (x_1 x_2 \cdots x_{2n})^{-p} \right)^{\frac{1}{2pn}} \right). \tag{2}$$

Here, the summation extends over all tuples of naturals $(x_1, \ldots, x_{2n})$ subject to the specified inequalities.

The following theorem is our main result. The sign $\prec$ in the summation index means $<$ or $\leq$ depending on the parity of the number n involved.

Theorem 2. Let $p > 1$ be a real number, and let

$$s(n) = \sum_{1 \le x_1 \le x_2 < x_3 \le x_4 < \ldots \prec x_n} (x_1 x_2 \cdots x_n)^{-p},$$

$$k_n = n(s(n))^{\frac{1}{np}}.$$

Then,

$$k_n \underset{n\to\infty}{\longrightarrow} \frac{e}{p} B\left(\frac{1}{2p}, 1 - \frac{1}{p}\right).$$

By Theorem 1 this assertion implies Valent's conjecture about type.

Before proceeding to the proof of Theorem 2, let us explain where does (2) come from, referring to [6] for details. The sums under the limit in there are essentially the Taylor coefficients of the Weierstraß product constructed from the spectrum of one of the self-adjoint extensions of L_p. Theorem 1 is obtained by substitution of these coefficients into the standard formula for the type of an entire function in terms of its Taylor coefficients.

Let us mention a wider context of the Valent conjecture. The cases where the order of a limit-circle Jacobi matrix is known are scarce. For instance, it can be shown [3] that the order of the matrix J with $a_i \equiv 0$ coincides with the convergence exponent of the sequence b_n^{-1} if $\sum (b_n b_{n-1})^{-1/2} < \infty$ and the sequence b_n is log-convex or log-concave. Some other classes of Jacobi matrices where the order is known are described in [5] and [6]. Apart from these, there are explicitly solvable orthogonal polynomials within the q–Askey scheme [10] that have order zero and has to be measured on a finer scale of growth.

Given the origins of the Valent conjecture in the analysis of the Kolmogorov equation, it would be interesting to get a probabilistic interpretation of the result.

It is worth to notice that the estimates obtained by the conventional variational approach (minimaximal principle) to the spectrum of the corresponding Jacobi matrices allow to give a nontrivial lower estimate for the type, but do not seem to be precise enough to calculate it.

Objects similar to $s(n)$ have appeared, see [14], in number-theoretical contexts as multiple zeta functions. The multiple zeta function is what we get if all the inequalities describing the range of indices in the definition of $s(n)$ are switched to the strict ones. The results available there do not seem to be useful in establishing the Valent conjecture, for it is easy to see that if the inequalities are switched to the strict ones, the

corresponding entire function has minimal type, that is, the limit of the corresponding k_n is 0.

The proof of Theorem 2 is decomposed into a dozen steps. To make the argument more digestible, the proofs of more technical ones are placed into a separate subsection called Details. The steps and their proofs in the Details subsection are enumerated accordingly. Throughout, C stands for all constants whose exact value is of no interest for us and may change from one line to another.

2 Proof of Theorem 2

2.1 Preliminaries

From now on, we fix the number p and do not indicate the dependence of various constants on it whenever reasonable.

Definition 1. Two sequences, x_n and y_n, of positive reals are said to be equivalent, denoting $x_n \approx y_n$, if $\ln(x_n/y_n) = o(n)$ as $n \to \infty$.

Obviously, $\approx$ is an equivalence relation, and if $nx_n^{\frac{1}{np}} \underset{n\to\infty}{\longrightarrow} a$ for some a then $ny_n^{\frac{1}{np}} \underset{n\to\infty}{\longrightarrow} a$ for any sequence y_n such that $y_n \approx x_n$. The proof will proceed by a successive reduction of the initial sequence to equivalent ones ending up with a sequence for which the required limit can be computed.

Before doing so, let us establish for future reference a warm-up estimate for $s(n)$.

Lemma 1. There exist $C_1, C_2 > 0$ such that

$$C_1^n n^{-pn} \le s(n) \le C_2^n n^{-pn}. \tag{3}$$

Let us first comment on the upper estimate. The summation indices, x_j, in the definition of $s(n)$ satisfy $x_j \ge \lfloor j/2 \rfloor$, hence $s(n)$ is estimated above by the sum over $x_j \ge \lfloor j/2 \rfloor$, which gives on account of the elementary inequality

$$\sum_{m=j+1}^{\infty} m^{-p} \le \frac{j^{1-p}}{p-1} \tag{4}$$

that for some C we have

$$s(n) \le C^n (\lfloor \frac{n}{2} \rfloor !)^{2(1-p)}.$$

This is worse than the actual upper estimate in (3) by the extra 2 in the exponent in the right-hand side. The bulk of the proof of the lemma consists in getting rid of this extra factor.

Proof. Observe first that taking $(x_1, x_2, \dots, x_n) = (1, 1, 2, 2, \dots)$ in the definition of $s(n)$ implies $s(2n) \geq (n!)^{-2p}$ and $s(2n-1) \geq ((n-1)!\, n!)^{-p}$ and the lower estimate in (3) follows.

To establish the upper estimate, it is sufficient to prove it for n even, for

$$s(2n+1) \leq \frac{(2n)^{1-p}}{p-1} s(2n) \tag{5}$$

by the argument mentioned after the formulation of the lemma. From now on we assume that n is even, hence, in particular, x_n takes values $x_{n-1}, x_{n-1}+1, \dots$. Applying (4) to the sum in x_n with $m = x_{n-1}$ we have,

$$s(n) \leq \sum_{1 \leq x_1 \leq x_2 < x_3 \leq x_4 < \dots \leq x_{n-2} < x_{n-1}} (x_1 x_2 \cdots x_{n-1})^{-p} \left[x_{n-1}^{-p} + \frac{x_{n-1}^{1-p}}{p-1} \right].$$

If $\lfloor \frac{n-1}{2} \rfloor > p-1$, then $x_{n-1} > p-1$ since $x_{n-1} \geq \lfloor \frac{n-1}{2} \rfloor$, and the sum in the square brackets in the right-hand side is estimated above by

$$\frac{2}{p-1} x_{n-1}^{1-p}.$$

Plugging this, we obtain that in the said range of n's

$$s(n) \leq \frac{2}{p-1} \sum_{1 \leq x_1 \leq x_2 < x_3 \leq x_4 < \dots \leq x_{n-2} < x_{n-1}} (x_1 x_2 \cdots x_{n-2})^{-p} x_{n-1}^{1-2p}.$$

Proceeding with the estimate, we have

$$s(n) \leq \frac{2}{2(p-1)^2} \sum_{1 \leq x_1 \leq x_2 < x_3 \leq x_4 < \dots < x_{n-3} \leq x_{n-2}} (x_1 x_2 \cdots x_{n-2})^{-p} x_{n-2}^{2-2p} \leq$$

$$\frac{2}{2(p-1)^2} \sum_{1 \leq x_1 \leq x_2 < x_3 \leq x_4 < \dots \leq x_{n-4} < x_{n-3}} (x_1 x_2 \cdots x_{n-3})^{-p} \left[x_{n-3}^{2-3p} + \frac{x_{n-3}^{3-3p}}{3(p-1)} \right].$$

If $x_{n-3} > 3(p-1)$ then

$$n^{n\alpha p \sum_{j=0}^{\frac{-\ln n}{\ln(1-\alpha)}} (1-\alpha)^j} (1-\alpha)^{\alpha p n (\sum_{j \geq 0} j(1-\alpha)^j + o(1))} = n^{-pn+O(1)} (1-\alpha)^{n(1+o(1))}.$$

Combining these we get the required upper estimate in (3):

$$x_{n-3}^{2-3p} + \frac{x_{n-3}^{3-3p}}{3(p-1)} \le 2x_{n-3}^{3-3p}. \tag{6}$$

Plugging this, one gets that

$$s(n) \le \frac{2\cdot 2}{2\cdot 3(p-1)^3} \sum_{1\le x_1\le x_2<x_3\le x_4<\ldots\le x_{n-4}<x_{n-3}} (x_1x_2\cdots x_{n-4})^{-p} x_{n-3}^{3-4p}$$

if $\frac{n}{2}-2 > 3(p-1)$ since $x_{n-3} \ge \lfloor\frac{n-3}{2}\rfloor = \frac{n}{2}-2$. Repeating this process, we get that if $\frac{n-j}{2} \ge j(p-1)$, j even, then

$$s(n) \le \frac{2^{1+j/2}}{j!\,(p-1)^j} \sum_{1\le x_1\le x_2<x_3\le x_4<\ldots\prec x_{n-j}} (x_1x_2\cdots x_{n-j-1})^{-p} x_{n-j}^{j-(j+1)p} \le$$

$$\frac{2^{1+j/2}}{j!\,(p-1)^j}\left(\frac{n-j}{2}\right)^{j(1-p)} s(n-j). \tag{7}$$

Thus, there exists a constant $C>0$ such that

$$s(n) \le \frac{C^n}{j^j}\,(n-j)^{j(1-p)}\,s(n-j),\ j\le \alpha n,\ \alpha = \frac{1}{2p-1}. \tag{8}$$

Here we account Stirling's formula to replace $j!$ in the denominator in the right-hand side of (7) by j^jC^j. It is convenient for us to define $s(t)$ for non-integer t by $s(t) = s(\lfloor t\rfloor)$. Then, the estimate (8) holds for non-integer n and j as well in the stated range of j's on account of (5). With $j=\alpha n$ it gives

$$s(n) \le \frac{C^n}{n^{p\alpha n}} s(n(1-\alpha)).$$

Applying this inequality successively $\lfloor\frac{-\ln n}{\ln(1-\alpha)}\rfloor$ times we find that

$$s(n) \le \frac{C^{n+n(1-\alpha)+n(1-\alpha)^2+\ldots}}{n^{p\alpha n}\,(n(1-\alpha))^{(1-\alpha)\alpha pn}\left(n(1-\alpha)^2\right)^{(1-\alpha)^2\alpha pn}\ldots}.$$

Since $\alpha<1$ the numerator is estimated from above by C^n for an appropriate C and the denominator is estimated from below by $n^{pn}C^n$ for small enough constant C. ■

2.2 Plan of the proof of the theorem

(1) Cutting the tails.

Lemma 2. For A large enough, the sequence

$$s'_n = \sum_{1\le x_1\le x_2<x_3\le\ldots\prec x_n\le T(n)} (x_1x_2\cdots x_n)^{-p},\ T(n)=n^A,$$

satisfies $s(n)\approx s'_n$.

(2) "Dyadisation".

Given an $\alpha>1$ ($\alpha-1$ is to be thought of as a small parameter in what follows) define a function P by $P(n)=\alpha^k$, $n\in[\alpha^k,\alpha^{k+1})$. Let $T'(n)=\alpha T(n)$, and

$$s'_n(\alpha) = \sum_{1\le x_1<x_2\le x_3<\ldots\prec x_n<T'(n)} (P(x_1)P(x_2)\cdots P(x_n))^{-p}, \tag{9}$$

$$S_n(\alpha) = \sum_{1\le x_1<x_2\le x_3<\ldots\prec x_n<T'(n)} (x_1x_2\cdots x_n)^{-p},$$

$$k'_n(\alpha)=n(s'_n(\alpha))^{\frac{1}{np}},\ K_n(\alpha)=n(S_n(\alpha))^{\frac{1}{np}}.$$

Since $T'(n)>T(n)$, we have $S_n(\alpha)\approx s(n)$ for every α. Then, the inequality

$$S_n(\alpha)\alpha^{-\frac{1}{np}}\le s'_n(\alpha)\le S_n(\alpha)$$

holds because $\frac{1}{\alpha}\le P(x)/x\le 1$. Thus, $s'_n(\alpha)\approx S_n(\alpha)$. The theorem will be established if we show that

$$k'_n(\alpha)\underset{n\to\infty}{\longrightarrow}\frac{e}{p}B\left(\frac{1}{2p},1-\frac{1}{p}\right)(1+O(\alpha-1)),\alpha\to 1$$

since the limit of left-hand side is independent of α.

(3) For a given $\alpha>1$ define $l(n)=\lfloor A\log_\alpha n\rfloor$. We will drop the argument α and the prime sign for notation convenience, writing $s_n=s'_n(\alpha)$ from now on. Each $P(x_i)$ in (5) is one of the numbers $1,\alpha,\ldots,\alpha^{l(n)}$. Let $c_i=\#\{y\colon P(y)=\alpha^i\}$, and let $H(a_0,a_1,\ldots,a_l)$

be the number of tuples $(x_1, x_2, \dots, x_n)$ such that $1 \le x_1 \le x_2 < \dots \prec x_n < T'(n)$ and $\#\{j\colon P(x_j) = \alpha^i\} = a_i$ for all i, $0 \le i \le l$. Then

$$s_n = \sum_{(a_0,a_1,\dots,a_l)\colon \sum a_i = n} H(a_0, a_1, \dots, a_n)\alpha^{p(-a_1-2a_2-\dots-la_l)}.$$

(4)

Lemma 3. Let $\mathrm{c}_0, \mathrm{c}_1, \dots, \mathrm{c}_l$ be nonnegative integers, and let $\mathrm{s}_0 = 0, \mathrm{s}_i = c_0+c_1+\dots+c_{i-1}$. Let $H(a_0, a_1, \dots, a_l)$ be the number of tuples $(x_1, x_2, \dots, x_n)$ such that $1 \le x_1 \le x_2 < \dots \prec x_n \le s_{l+1}$ and $\#\{j :\ s_i < x_j \le s_{i+1}\} = \mathrm{a}_i$ for all i, $0 \le i \le l$. Let also

$$\mathrm{w}_i = \begin{cases} \lfloor \frac{a_i-1}{2} \rfloor, & \sum_{j=0}^{i-1} \mathrm{a}_j \ odd \\ \lceil \frac{a_i-1}{2} \rceil, & otherwise. \end{cases}$$

Then

$$H(\mathrm{a}_0, a_1, \dots, a_l) = \prod_{i=0}^{l} \binom{c_i + w_i}{a_i}.$$

We apply this Lemma to a_i, c_i, H from (3) and deduce that

$$H(a_0, a_1, \dots, a_l) = \prod_{i=0}^{l} \binom{c_i + w_i}{a_i}$$

with w_i defined as in Lemma 3.

(5) Replacing the product by its maximum and passing to the continuous problem. Let $\Pi_n = \{\vec{x} := (x_0, \dots, x_l) \in \mathbb{R}^{l+1}\colon 0 \le x_j \le 2c_j, \sum x_j = n\}$, and let

$$s_n^{(4)} = \max_{\vec{x}\in\Pi_n} \prod_{i=0}^{l} \exp\left(-ix_i p \ln\alpha + \left(\frac{x_i}{2} + c_i\right)\ln\left(\frac{x_i}{2} + c_i\right) - x_i \ln x_i - \left(c_i - \frac{x_i}{2}\right)\ln\left(c_i - \frac{x_i}{2}\right)\right). \tag{10}$$

Then $s_n^{(4)} \approx s_n$.

(6) Calculating the maximum. Let us denote by G the argument of the exponent in (10), that is,

$$G = \sum_{i=0}^{l} -ix_i p \ln\alpha + \left(\frac{x_i}{2} + c_i\right)\ln\left(\frac{x_i}{2} + c_i\right) - x_i \ln x_i - \left(c_i - \frac{x_i}{2}\right)\ln\left(c_i - \frac{x_i}{2}\right).$$

Maximizing $s_n^{(4)}$ is the same that maximizing G. Consider the conditional extremal problem for G on the hyperplane $\sum x_j = n$. Notice that the derivatives $\partial G/\partial x_i$ are all equal at a critical point. On calculating,

$$\frac{\partial G}{\partial x_i} = -ip\ln\alpha + \frac{\ln(\frac{x_i}{2}+c_i)}{2} - \ln x_i + \frac{\ln(c_i-\frac{x_i}{2})}{2} = -ip\ln\alpha + \frac{1}{2}\ln\left(\frac{c_i^2}{x_i^2}-\frac{1}{4}\right).$$

The right-hand side is a monotone decreasing function of x_i on the interval $(0, 2c_i)$, going to $+\infty$ as $x_i \to 0^+$, to $-\infty$ as $x_i \to 2c_i^-$, hence the equation $\frac{\partial G}{\partial x_i} = \lambda$ has exactly one solution,

$$x_j(\lambda) = \frac{2c_i}{\sqrt{4\alpha^{2ip}e^{2\lambda}+1}}.$$

It follows that there is exactly one critical point, which is determined from the equation $\sum x_i(\lambda) = n$, and it is easy to see that this point is the required point of maximum. On plugging this maximum into (6) we get

$$s_n^{(4)} = \exp\left(\lambda n + \sum_{i=0}^{l} c_i \ln\frac{\sqrt{D_i}+1}{\sqrt{D_i}-1}\right), \tag{11}$$

$$D_i := 1 + 4\alpha^{2ip}e^{2\lambda}.$$

(7)

Lemma 4. The equation

$$\sum_{i=0}^{l} \frac{2c_i}{\sqrt{4\alpha^{2ip}e^{2\lambda}+1}} = n \tag{12}$$

implies the following asymptotics of λ as $n \to \infty$,

$$\limsup_{n\to\infty}\left|\lambda - \left(-p\ln n + p\ln\left[\frac{\alpha-1}{\ln\alpha}J(p)\right]\right)\right| = O(\alpha-1), \tag{13}$$

$$J(p) = 2^{1-1/p}\int_0^\infty \frac{du}{\sqrt{1+u^{2p}}}.$$

(8) Plugging the asymptotics (13) into (11) we have,

$$n\left(s_n^{(4)}\right)^{\frac{1}{np}} = \frac{\alpha-1}{\ln\alpha}J(p)\exp\left(\frac{\xi}{np} + o(1) + O(\alpha-1)\right), \tag{14}$$

$$\xi := \sum_{i=0}^{l} c_i \ln\frac{\sqrt{D_i}+1}{\sqrt{D_i}-1}.$$

(9) For each $\alpha > 1$ the quantity ξ satisfies

$$n\frac{1-\alpha^{-2p}}{2(\alpha-1)} + o(1) \le \xi \le n\alpha\frac{\alpha^{2p}-1}{2(\alpha-1)} + o(1)$$

as $n \to \infty$. On plugging this in (14) we find

$$\limsup_{n\to\infty} n\left(s_n^{(4)}\right)^{\frac{1}{np}} = \frac{\alpha-1}{\ln\alpha} eJ(p)q(\alpha),$$

$$q(\alpha) = 1 + O(\alpha - 1).$$

Passing to the limit $\alpha \to 1$ we establish that

$$\limsup_{n\to\infty} n\left(s(n)\right)^{\frac{1}{np}} = eJ(p).$$

It now remains to notice that the substitution $v = \left(1+u^{2p}\right)^{-1}$ in the definition of $J(p)$ gives

$$J(p) = \frac{1}{p}2^{-\frac{1}{p}}B\left(\frac{1}{2p}, \frac{1}{2} - \frac{1}{2p}\right) = \frac{1}{p}B\left(\frac{1}{2p}, 1 - \frac{1}{p}\right)$$

on account of the identity $\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)$, see, for example, [8]. The theorem is established.

2.3 Details (1)

Proof of Lemma 2 Let us decompose the set of tuples $(x_1, x_2, \ldots, x_n)$ over which the summation in $s(n)$ takes place into $n+1$ disjoint sets according to the number of x_j's in a given tuple, which are greater than $T(n)$, and split the sum accordingly. More precisely, let

$$\mathcal{M}_n = \{(x_1, x_2, \ldots, x_n)\colon 1 \le x_1 \le x_2 < x_3 \le \ldots \prec x_n\},$$

$$\mathcal{M}_n^k = \{(x_1, x_2, \ldots x_n) \in \mathcal{M}_n\colon \#\{j : x_j > T(n)\} = k\},\ 0 \le k \le n.$$

Denote by $s_n^{(k)}$ the sums over $\mathcal{M}_n^k$. Then

$$s(n) = \sum_{k=0}^{n} s_n^{(k)}.$$

We are going to show that s_n and $s_n^{(0)} \equiv s'_n$ are equivalent by establishing the following two assertions.

Claim 1: If n is large enough, then $s_n^{(k+2)} \le s_n^{(k)}$ for all k;

Claim 2: $s_n^{(1)} \le Cn^{C_A} s_n^{(0)}$ for some constants C, C_A, depending on A.

Proof of Claim 1.

Notice that for any $\vec{x} := (x_1, x_2, \dots, x_n) \in \mathcal{M}_n^k$ there exists a (possibly, nonunique) integer $m \le 2n+4$ such that neither m, nor $m+1$, are among the x_j's. For each $\vec{x} \in \mathcal{M}_n^{k+2}$ fix such an integer $m = m(\vec{x})$ in an arbitrary way. Consider the following transformation of the tuple $\vec{x}$,

Replace x_{n-1} and x_n by m and $m+1$, respectively, and rearrange the new sequence in the ascending order.

The resulting tuple obviously belongs to $\mathcal{M}_n^k$ for n large enough. Let $F\colon \mathcal{M}_n^{k+2} \to \mathcal{M}_n^k$ stand for the map just defined. Given a $\vec{t} = (t_1, \dots t_n) \in \mathcal{M}_n^k$ let us define

$$\mathcal{M}_{\vec{t}} = \left\{ \vec{x} \in \mathcal{M}_n^{k+2} \colon F(\vec{x}) = \vec{t} \right\}.$$

We then have

$$\sum_{\mathcal{M}_{\vec{t}}} (x_1 \dots x_n)^{-p} = \sum_{l=1}^{n} \sum_{\vec{x} \in \mathcal{M}_{\vec{t}} \colon m(\vec{x}) = t_l} \left(x_1 \dots x_n\right)^{-p}. \tag{15}$$

In the inner sum in the right-hand side $x_j = t_j$ for $j < l$; $x_j = t_{j+2}$ for $l \le j \le n-2$, hence the summation actually takes place in x_{n-1} and x_n only. Let us estimate it taking into account that $x_n \ge x_{n-1} > T(n)$ for $\vec{x} \in \mathcal{M}_n^{k+2}$, which gives

$$\sum_{\vec{x} \in \mathcal{M}_{\vec{t}} \colon m(\vec{x}) = t_l} \left(x_1 \dots x_n\right)^{-p} \le CT(n)^{2(1-p)} \left(t_1 \cdots t_{l-1} t_{l+2} \cdots t_n\right)^{-p} \tag{16}$$

Then

$$\left(t_1 \cdots t_{l-1} t_{l+2} \cdots t_n\right)^{-p} = m^p (m+1)^p (t_1 \cdots t_n)^{-p} \le Cn^{2p} (t_1 \cdots t_n)^{-p}.$$

Plugging this estimate into (16), we find that

$$\sum_{\vec{x} \in \mathcal{M}_{\vec{t}} \colon m(\vec{x}) = t_l} \left(x_1 \dots x_n\right)^{-p} \le Cn^{2A(1-p)+2p} (t_1 \cdots t_n)^{-p}.$$

The right-hand side does not depend on l, hence the sum in the left-hand side in (15) is estimated as

$$\sum_{\mathcal{M}_{\vec{t}}} (x_1 \dots x_n)^{-p} \le Cn^{2A(1-p)+2p+1} (t_1 \cdots t_n)^{-p}.$$

The factor in front of the product of t_j's is < 1 for n large enough provided that $A > \frac{2p+1}{2(p-1)}$. Fixing an arbitrary such A and summing over $\vec{t} \in \mathcal{M}_n^k$ we obtain that $s_n^{(k+2)} \leq s_n^{(k)}$, as required.

Proof of Claim 2.

It is clear that the map F defined above acts from $\mathcal{M}_n^1$ to $\mathcal{M}_n^0$. We proceed as in the proof of Claim 1 up until the estimate (16). In the situation under consideration $x_n > T(n)$, but $x_{n-1} \leq T(n)$, hence we are going to have

$$\sum_{\vec{x}\in\mathcal{M}_l:\ m(\vec{x})=t_l} \left(x_1 \dots x_n\right)^{-p} \leq CT(n)^{(1-p)} \left(t_1 \cdots t_{l-1} t_{l+2} \cdots t_n\right)^{-p},$$

instead of (16). Plugging this into (15), we obtain that $s_n^{(1)} \leq Cn^{A(1-p)+2p+1} s_n^{(0)}$, as required. ■

(4)

Proof of Lemma 3. The integers from 1 to s_{l+1} are split into $l+1$ groups of respective sizes $c_0, c_1, \dots, c_l$, and according to the definition of H, there are a_i numbers chosen from the i-th group. With $a_0, a_1, \dots, a_l$ fixed the choices from different groups are independent, so $H(a_0, a_1, \dots, a_n)$ is a product of numbers of ways to choose a_i numbers, $\{y_j\}_{j=1}^{a_i}$, such that $y_1 \prec y_2 \dots \prec y_{a_i}$ from the i-th group. Consider i-th group. Without loss of generality one can shift the enumeration of y_j's so that they become integers from the interval $[1, c_i]$. Let us consider the following transformation of the sequence $\{y_j\}$,

$$(y_1, y_2 \dots, y_{a_i}) \to (y_1 + m_1, y_2 + m_2, \dots, y_{a_i} + m_{a_i}),$$

where

$$m_i = \begin{cases} \lfloor \frac{i-1}{2} \rfloor, & \sum_{j=0}^{i-1} a_j \ odd \\ \lceil \frac{i-1}{2} \rceil, & otherwise. \end{cases}$$

Notice that $w_i = m_{a_i}$. This is a bijective transformation between sequences $\{y_j\}_1^{a_i}$ subject to $y_1 \prec y_2 \dots \prec y_{a_i}$ and strictly increasing sequences $x'_1 < x'_2 < \dots < x'_{a_i} \leq m_{a_i} + c_i$. Thus, the number of choices for i-th group is equal to the number of strictly increasing sequences $x'_1 < x'_2 < \dots < x'_{a_i} \leq w_i + c_i$, and the latter is $\binom{c_i+w_i}{a_i}$. Thus,

$$H(a_0, a_1, \dots, a_l) = \prod_{i=0}^{l} \binom{c_i + w_i}{a_i},$$

as required. ■

(5.1) At first, we can replace w_i with $\frac{a_i}{2}$. Namely, let

$$H'(a_0, a_1, \dots, a_l) = \prod_{i=0}^{l} \alpha^{-ia_ip} \binom{c_i + \frac{a_i}{2}}{a_i} \tag{17}$$

and

$$s_n^{(1)} = \sum_{(a_0,a_1,\dots,a_l):\ \sum a_i = n} H'(a_0, a_1, \dots, a_n).$$

Then $s_n^{(1)} \approx s_n$. This is cause

$$(2T'(n))^2 \binom{c_i + w_i}{a_i} \geq \binom{c_i + \frac{a_i}{2}}{a_i} \geq \frac{\binom{c_i+w_i}{a_i}}{(2T'(n))^2},$$

and thus the ratio of values of H and H' on the same set of a_j's satisfies

$$(2T'(n))^{2l+2} \geq \frac{H}{H'} \geq \frac{1}{(2T'(n))^{2l+2}}.$$

Since $2T'(n))^{2l+2} = e^{O(\ln^2 n)}$ this implies the assertion of this item.

(5.2) Then we can notice that the sum can be replaced by the maximum of the summand, so if we denote

$$s_n^{(2)} = \max_{(a_0,a_1,\dots,a_l):\ \sum a_i = n} H'(a_0, a_1, \dots, a_n).$$

Then $s_n^{(2)} \approx s_n^{(1)}$. The assertion follows from the fact that the number of terms in the sum does not exceed the number of subsets of $l+1$ integers lying between 0 and n, which is $(n+1)^{l+1} = e^{O(\ln^2 n)} = e^{o(n)}$.

(5.3) We can replace the maximum over an integer hyperplane by the maximum over the real one. Notice that the right-hand side in the definition (17) can be rewritten as

$$H'(a_0, a_1, \dots, a_l) = \prod_{i=0}^{l} \frac{\alpha^{-ia_ip}}{(c_i + \frac{a_i}{2} + 1)B(a_i + 1, c_i - \frac{a_i}{2} + 1)},$$

which makes sense for all real a_j, $0 \leq a_i \leq 2c_i$. Let $\Pi_n = \{(x_0, \dots, x_l) \in \mathbb{R}^{l+1} : 0 \leq x_j \leq 2c_j, \sum x_j = n\}$, and

$$s_n^{(3)} = \max_{\Pi_n} H'(x_0, x_1, \dots, x_l).$$

Then $s_n^{(3)} \approx s_n^{(2)}$.

First, $s_n^{(3)} \geq s_n^{(2)}$ as maximum in the definition of $s_n^{(3)}$ is taken over a larger set. Second, let $(X_0, X_1, \ldots, X_l)$ be the point of maximum of H', and let $Y_i = \lceil X_i \rceil$. We proceed in two steps:

Step 1: Replace all of X_i by Y_i, and let $M = \sum Y_i - \sum X_i$. Notice that $M = O(\ln n)$. Let us write explicitly the expression for the maximal value of H',

$$H'(X_0, X_1, \ldots, X_l) = \prod_{i=0}^{l} \frac{\alpha^{-iX_i p} \Gamma\left(\frac{X_i}{2} + c_i + 1\right)}{\Gamma\left(\frac{-X_i}{2} + c_i + 1\right) \Gamma(X_i + 1)}.$$

When we replace X_i by Y_i, the value of $\alpha^{-iX_i p}$ will change by an $O(T'(n)^p)$ multiple, and the γ-functions - by $O(T'(n))$ multiples, so every factor will gain an at most an $e^{O(T'(n))}$ multiple, thus the whole product will change by no more than an $e^{O(\ln^2 n)} = e^{o(n)}$ multiple.

Step 2: Replace any nonzero Y_i by $Y_i - 1$. Again every operation will change the product in at most $e^{O(T'(n))}$ times, and the number of operations is $O(\ln n)$.

After these two steps, we will obtain a set of integers, $(X_0', X_1', \ldots, X_n')$ subject to $\sum_j X_j' = n$, for which

$$e^{o(n)} \geq \frac{H'(X_0', X_1', \ldots, X_n')}{H'(X_0, X_1, \ldots, X_n)} \geq e^{-o(n)},$$

and the assertion follows.

(5.4) Then we apply the Stirling's formula, for every $x \leq n$ we have

$$e^{-O(\ln(n))} \leq \frac{e^{x \ln(x) - x}}{\Gamma(x+1)} \leq e^{O(\ln(n))},$$

so when

$$\frac{\alpha^{-ia_i p}}{(c_i + \frac{a_i}{2} + 1) B(a_i + 1, c_i - \frac{a_i}{2} + 1)}$$

is replaced by

$$\exp\left(-ia_i p \ln \alpha + \left(\frac{a_i}{2} + c_i\right) \ln\left(\frac{a_i}{2} + c_i\right) - a_i \ln a_i - \left(c_i - \frac{a_i}{2}\right) \ln(c_i - \frac{a_i}{2})\right)$$

the product multiplies by an $e^{O(\ln(n))}$ factor, hence

$$\prod_{i=0}^{l} \frac{\alpha^{-ia_i p}}{(c_i + \frac{a_i}{2} + 1) B(a_i + 1, c_i - \frac{a_i}{2} + 1)} =$$

$$e^{o(n)} \prod_{i=0}^{l} \exp\left(-\ln(\alpha) i a_i p + (\frac{a_i}{2} + c_i) \ln(\frac{a_i}{2} + c_i) - a_i \ln(a_i) - (c_i - \frac{a_i}{2}) \ln(c_i - \frac{a_i}{2})\right),$$

and the assertion follows.

(7)

Proof of lemma 4. First, notice that (see (12))

$$n = \sum_{j=0}^{l} \frac{2c_j}{\sqrt{D_j}} \leq (\alpha - 1)e^{-\lambda} \sum_{j=0}^{l} \alpha^{i(1-p)} \leq C(\alpha)e^{-\lambda}. \tag{18}$$

Then, observe that $D_l \to \infty$ as $n \to \infty$. Indeed, otherwise the sum $\sum_{j=0}^{l} \frac{c_i}{\sqrt{D_i}}$ is estimated below by $n^l \geq Cn^A$ on a sequence of n for a non-zero C. Fix an $M > 0$ and let n be large enough so as to there exists a $j_M < l$ such that $D_j > M$ for $j \geq j_M$, $D_j \leq M$ for $j < j_M$. Clearly, $l - j_M \to \infty$ as $n \to \infty$ hence for all n large enough we are going to have,

$$\sum_{j=0}^{l} \frac{2c_i}{\sqrt{D_i}} \geq \sum_{j=j_M}^{l} (\dots) \geq e^{-\lambda}(\alpha - 1) \sum_{j=j_M}^{l} \alpha^{i(1-p)} = e^{-\lambda} \frac{\alpha - 1}{1 - \alpha^{1-p}} \alpha^{j_M(1-p)} (1 + o(1)) \geq$$
$$C_\alpha e^{-\lambda - \lambda \frac{1-p}{p}} (1 + o(1)).$$

Here, C_α is a constant in n uniformly separated from zero for small $\alpha - 1$. Thus, there exists a positive C for all α close enough to 1

$$\limsup_{n \to \infty} \frac{e^{-\lambda/p}}{n} < C. \tag{19}$$

Proceeding, notice that

$$\sum_{j=0}^{l} \frac{2\alpha^j}{\sqrt{1 + 4\alpha^{2jp} e^{2\lambda}}} = \int_0^l \frac{2\alpha^x \mathrm{d}x}{\sqrt{1 + 4\alpha^{2px} e^{2\lambda}}} + O\left(e^{-\lambda/p}\right).$$

To establish this, it is enough to notice that the function $h(s) = \frac{s}{\sqrt{s^{2p}+1}}$ has two regions of monotonicity over $\mathbb{R}_+$ separated by a point of maximum. It follows that the difference of the sum and the integral has absolute value not greater than three times the maximum of the integrand, and that the maximum is estimated above by $Ce^{-\lambda/p}$ with the positive constant p depending on p only. On substituting, this and (19) give

$$\limsup_{n \to \infty} \left| 1 - (\alpha - 1) \frac{1}{n} \int_0^l \frac{2\alpha^x \mathrm{d}x}{\sqrt{1 + 4\alpha^{2px} e^{2\lambda}}} \right| = O(\alpha - 1).$$

On the other hand,

$$\int_0^l \frac{2\alpha^x e^{2\lambda} \mathrm{d}x}{\sqrt{1 + 4\alpha^{2px}}} = e^{-\lambda/p} \frac{1}{\ln \alpha} \int_{e^{\lambda/p} 2^{1/p}}^{\alpha^l 2^{1/p} e^{\lambda/p}} \frac{2^{1-1/p} \mathrm{d}u}{\sqrt{1 + u^{2p}}}.$$

The lower limit of integration vanishes in the limit $n \to \infty$ by (18), the upper one goes to infinity by (19) and the fact that $\alpha^l \asymp n^A$ with an $A > 1$, hence the integral in the right-hand side goes to $J(p)$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} \left| 1 - \frac{\alpha - 1}{\ln \alpha} \frac{e^{-\lambda/p}}{n} (J(p) + o(1)) \right| = O(\alpha - 1),$$

which is (13). ■

(9) Applying the summation by parts, we find

$$\xi = \alpha^{l+1} \ln \frac{\sqrt{D_l} + 1}{\sqrt{D_l} - 1} + \sum_{j=0}^{l} \alpha^j \left[\ln \frac{\sqrt{D_{i-1}} + 1}{\sqrt{D_{i-1}} - 1} - \ln \frac{\sqrt{D_i} + 1}{\sqrt{D_i} - 1} \right].$$

The 1st term in the right-hand side is $O(\alpha^{l(1-p)} e^{-\lambda}) = O(n^{A(1-p)+p})$ and hence vanishes as $n \to \infty$ with our choice of A. The 2nd term is estimated by applying the mean value theorem to the difference in the square brackets, which gives

$$[\dots] = \alpha^{2(i-i_*)p} \frac{1 - \alpha^{-2p}}{\sqrt{1 + 4\alpha^{2i_* p} e^{2\lambda}}}$$

for some $i_* \in (i-1, i)$. Replacing the denominator with $\sqrt{D_i}$, then $\sqrt{D_{i-1}}$, and taking into account that $\sum_{j=0}^{l} \frac{2c_i}{\sqrt{D_i}} = n$, we obtain

$$n \frac{1 - \alpha^{-2p}}{2(\alpha - 1)} \le \sum^{l} \alpha^j \left[\ln \frac{\sqrt{D_{i-1}} + 1}{\sqrt{D_{i-1}} - 1} - \ln \frac{\sqrt{D_i} + 1}{\sqrt{D_i} - 1} \right] \le n\alpha \frac{\alpha^{-2p} - 1}{2(\alpha - 1)},$$

and the assertion follows.

3 A Slight Extention

We can generalize the problem in the following way: let

$$s(n) = \sum_{1 \le x_1 \le x_2 < x_3 \le x_4 < \dots < x_n} F(x_1) F(x_2) \cdots F(x_n), k_n = n(s(n))^{\frac{1}{np}}.$$

Our aim is to calculate $K = \limsup k_n$. Result of this paper refer to the special case $F(x) = x^{-p}$.

The above proof uses some specific properties of the growth function x^p in the model problem; however, the initial steps in the argument admit some generalization. More precisely, the following assertion holds.

Theorem 3. Let F be monotone nonnegative increasing function in $[0,\infty]$, such that $\frac{d\ln F}{d\ln x}$ exists and is nonnegative everywhere. Moreover, let p_1, p_2 be constants, such that everywhere $p_1 < \frac{d\ln F}{d\ln x} < p_2$, and let the notation K, s, Π_n be as above. Define

$$s_n^{(4)} = \max_{\Pi_n} \prod_{i=0}^{l} \exp\left(-a_i \ln F(\alpha^i) + \left(\frac{a_i}{2} + c_i\right) \ln\left(\frac{a_i}{2} + c_i\right) - a_i \ln a_i - \left(c_i - \frac{a_i}{2}\right) \ln\left(c_i - \frac{a_i}{2}\right)\right), \tag{20}$$

where the tuple $(a_0, a_1, \ldots, a_l) \in \Pi_n$. Then there exist a constant A, depending only at p_1, p_2, such that for $l = \lceil A \log(n) \rceil$ we have

$$K = \limsup n s_1(n)^{\frac{1}{np}}.$$

The proof differs in very little details, so let us briefly go through the items and explain similarity:

(1) $C_1 n^{p_1} < F(n) < C_2 n^{p_2}$ for some constants C_1, C_2. This means that $C_2^n n^{-p_2 n} < F(n) < C_1^n n^{-p_1 n}$ for some constants C_1, C_2 by Lemma 1. Then we can literally repeat the proof of the 1st item and obtain that for A large enough the sequence

$$s'_n = \sum_{1 \le x_1 \le x_2 < x_3 \le \ldots \prec x_n < T(n)} F(x_1)F(x_2)\cdots F(x_n),\ T(n) = n^A,$$

satisfies $s(n) \approx s'_n$.

(2)

$$s'_n(\alpha) = \sum_{1 \le x_1 < x_2 \le x_3 < \ldots \prec x_n < T'(n)} F(P(x_1))F(P(x_2))\cdots F(P(x_n)). \tag{21}$$

The proof repeats the one of the corresponding step in Theorem 2.

(3)

$$s_n = \sum_{(a_0,a_1,\ldots,a_l):\ \sum a_i = n} H(a_0, a_1, \ldots, a_n) P(\alpha^0)^{-a_0} P(\alpha^1)^{-a_1} \ldots P(\alpha^l)^{-a_l}.$$

The proof repeats one of the corresponding steps in Theorem 2. Proofs of steps (4) and (5) can be taken verbatim from Theorem 2.

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