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Integral Equations and Operator Theory

Mate–Nevai–Totik Theorem for Krein Systems

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Abstract. We prove the Cesàro boundedness of eigenfunctions of the Dirac operator on the half-line with a square-summable potential. The proof is based on the theory of Krein systems and, in particular, on the continuous version of a theorem by A. Máté, P. Nevai and V. Totik from 1991.

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1. Introduction

1.1. Main Result

Consider a symmetric zero trace 2×2 potential $Q = \begin{pmatrix} -q & p \\ p & q \end{pmatrix}$ on $\mathbb{R}_+ = [0, +\infty)$ with real-valued entries. We say that $Q \in L^r(\mathbb{R}_+)$ if both functions $p, q \in L^r(\mathbb{R}_+)$. Consider the differential equation

$$Jf'(t,\lambda) + Qf(t,\lambda) = \lambda f(t,\lambda), \quad t \in \mathbb{R}_+, \quad \lambda \in \mathbb{C}, \quad f(0,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.1}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a square root of the minus identity matrix. This equation can be read as eigenfunction equation for Dirac operator $D_Q = J\frac{d}{dr} + Q$ on the half-line $\mathbb{R}_+$.

In 1955, M. Krein proved [16] that the absolutely continuous spectrum of D_Q is the whole real line $\mathbb{R}$ in the case when $Q \in L^2(\mathbb{R}_+)$. In 1999, P. Deift and R. Killip showed [7] that the absolutely continuous spectrum of one-dimensional Shrödinger operator with L^2 potential fills the half-line

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$\mathbb{R}_+$. These results were later strengthened by M. Christ and A. Kiselev who proved [4–6] the existence of modified Jost solutions and wave operators for Shrödinger and Dirac operators with potentials in L^r, $1 \leqslant r < 2$, and by Denisov [9] who established the existence of wave operators for Dirac operators with potentials in $L^2(\mathbb{R}_+)$. Recently R. Bessonov extended [2] this result for Dirac operators whose spectral measure belongs to the Szegő class on the real line. Some related results and approaches can be found in [19,20, 23].

M. Christ and A. Kiselev proved in Section 6.3 in [6] that the solution of (1.1) is bounded for almost every $\lambda \in \mathbb{R}$ in the case when $Q \in L^r(\mathbb{R}_+)$, $1 \leqslant r < 2$. Related question was studied by C. Muscalu, T. Tao, and C. Thiele in [19]; the authors posed the following conjecture.

Conjecture I. *Assume that $Q \in L^2(\mathbb{R}_+)$. Then for almost every $\lambda \in \mathbb{R}$ the solution f of (1.1) is bounded on $\mathbb{R}_+$.*

In the present paper we prove an "averaged" version of Conjecture I.

Theorem 1. *Assume that $Q \in L^2(\mathbb{R}_+)$. Then for almost every $\lambda \in \mathbb{R}$ the solution f of (1.1) is bounded on $\mathbb{R}_+$ in the L^2-Cesàro sense, i.e.,*

$$\sup_{r>0} \left(\frac{1}{r} \int_0^r |f(t,\lambda)|^2 \, dt \right) < \infty.$$

The proof of this theorem is based on the theory of Krein systems which plays a role of a bridge between the theory of orthogonal polynomials on the unit circle and spectral theory of second order differential self-adjoint operators. Krein systems were first introduced by M. Krein in [16] and later developed by different authors (see, e.g., [3,8,21,22,26]). Detailed account of the theory can be found in [10].

Let us start with a description of basic objects and results in the theory of orthogonal polynomials and in the theory of Krein systems.

1.2. Orthogonal Polynomials on the Unit Circle

Orthogonal polynomials on the unit circle appear in many areas of mathematics. An account of the theory can be found in books [25] by G. Szegő and [24] by B. Simon. Let μ be a probability measure on $[-\pi, \pi]$ such that the support of μ is not a finite set. Then there exists an infinite family of orthonormal polynomials $\{\varphi_n\}_{n \geqslant 0}$ in $L^2([-\pi,\pi], \mu)$. More precisely, let the family φ_n be defined by

$$\deg \varphi_n = n, \quad \langle \varphi_n, \varphi_m \rangle_{L^2(\mu)} = \int_{-\pi}^{\pi} \varphi_n(e^{it}) \overline{\varphi_m(e^{it})} \, d\mu(t) = \delta_{n,m}, \quad (1.2)$$

where $\delta_{n,m}$ is the Kronecker symbol. Define a family of reversed orthogonal polynomials by

$$\varphi_n^*(z) = z^n \overline{\varphi_n(1/\overline{z})}, \quad z \in \mathbb{C}.$$

It is easy to see that φ_n^* is a polynomial of degree not greater that n. We choose φ_n so that $\varphi_n^*(0) > 0$. We say that a measure μ belongs to the Szegő

class on the unit circle if

$$\int_{-\pi}^{\pi} \log \mu'(t)\, dt > -\infty, \tag{1.3}$$

where μ' is the density of the absolutely continuous part of μ with respect to the Lebesgue measure on $[-\pi, \pi]$. The following theorem is a result of work of different authors, we refer to it as the Szegő theorem. For the proof, combine Theorem 2.4.1 and Theorem 2.7.15 in [24].

Theorem A. *(Szegő theorem) Let μ be a nontrivial measure on $[-\pi, \pi]$ and let $\{\varphi_n\}_{n\geqslant 0}$ be the associated orthogonal polynomials. Then the following assertions are equivalent:*

(*a*) *the measure μ belongs to the Szegő class,*

(*b*) *for some $z_0 \in \mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ we have $\sum_{n=0}^{\infty} |\varphi_n(z_0)|^2 < \infty$,*

(*c*) *there exists an analytic in $\mathbb{D}$ function Π such that*

$$\Pi(z) = \lim_{n\to\infty} \varphi_n^*(z), \quad z \in \mathbb{D},$$

and the convergence is uniform on compact subsets of $\mathbb{D}$.

If the equivalent assertions of the Szegő theorem hold, then the series in part (*b*) converges uniformly on compact subsets of $\mathbb{D}$. Moreover, the function Π is an outer function in the unit disk (see Theorem 2.3.5 in [24]) and $|\Pi(e^{it})|^{-2} = \mu'(t)$ almost everywhere on $[-\pi, \pi]$, i.e.,

$$\Pi(z) = \exp\left[-\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log \mu'(t)\, dt\right], \quad z \in \mathbb{D}.$$

The function $D = \Pi^{-1}$ is called the Szegő function and is often used instead of Π.

1.3. Krein Systems

Krein systems were introduced by Krein [16] in 1954. Solutions of Krein systems have properties similar to properties of orthogonal polynomials on the unit circle. Through such similarities it becomes possible to apply methods and ideas from the theory of orthogonal polynomials on the unit circle to spectral problems for self-adjoint differential operators. Detailed account of this approach can be found in the paper [10] by S. Denisov.

Definition. Let a be a complex-valued function on the half-line $\mathbb{R}_+$. The Krein system with the coefficient a is the following system of differential equations:

$$\begin{cases} \frac{\partial}{\partial r} P(r,\lambda) = i\lambda P(r,\lambda) - \overline{a(r)} P_*(r,\lambda), & P(0,\lambda) = 1, \\ \frac{\partial}{\partial r} P_*(r,\lambda) = -a(r) P(r,\lambda), & P_*(0,\lambda) = 1. \end{cases} \tag{1.4}$$

Here the complex number λ is a spectral parameter; solutions P and P_* are called continuous analogs of orthogonal polynomials on the unit circle. In the present paper we consider the case when $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, that is, $a \in L^1[0, x]$ for every $x \geqslant 0$.

Solutions $P(r,\cdot)$ and $P_*(r,\cdot)$ are entire functions. The parameter $r \geqslant 0$ plays the same role as the index $n \geqslant 0$ in (1.2): $P(r,\cdot)$ has the exponential type r and $P_*(r,\cdot)$ has the exponential type not greater than r, see Chapter 3 in [10]. For any function $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ there exists a Borel measure σ on the real line (see [10]) such that

$$\int_{\mathbb{R}} \frac{d\sigma(x)}{1+x^2} < \infty, \tag{1.5}$$

and the map

$$\mathcal{U}_\sigma : f \mapsto \int_0^\infty f(r)P(r,\lambda)\,dr,$$

initially defined on simple measurable functions with compact support, can be continuously extended to an isometry from $L^2(\mathbb{R}_+)$ to $L^2(\mathbb{R},\sigma)$. This measure is called the spectral measure of system (1.4).

The paper [10] contains the discussion of the case when $a \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, however the argument provided in Section 8 in [10] is based only on the existence of spectral measure of Krein system and on Christoffel–Darboux formula. Both of this facts are proved in [21] for case when $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, hence the results in Section 8 in [10] hold in our case. Below in the text we will refer to [10] with this remark in mind.

We say that the measure σ belongs to the Szegő class on the real line if it satisfies condition (1.5) and

$$\int_{\mathbb{R}} \frac{|\log \sigma'(x)|}{1+x^2}\,dx < \infty, \tag{1.6}$$

where σ' is the density of the absolutely continuous part of σ with respect to the Lebesgue measure on $\mathbb{R}$. A key result in the theory of the Krein systems is an analog of the Szegő theorem. It was originally formulated by Krein in [16] with minor inaccuracies subsequently corrected by A. Teplyaev in [26]. For a complete proof, see Chapter 8 in [10].

Theorem B. (Krein theorem) *Let a measure σ be the spectral measure of some Krein system (1.4) and let P, P_* be the solutions of (1.4). Then the following assertions are equivalent:*

(a) *the measure σ belongs to Szegő class,*
(b) *for a point λ_0 in $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ we have*

$$\int_0^\infty |P(r,\lambda_0)|^2\,dr < \infty,$$

(c) *there exists analytic in $\mathbb{C}_+$ function Π and a sequence $r_n \to +\infty$ such that*

$$\Pi(\lambda) = \lim_{n\to\infty} P_*(r_n,\lambda), \quad \lambda \in \mathbb{C}_+,$$

and the convergence is uniform on compact subsets of $\mathbb{C}_+$.

If the equivalent assertions of the Krein theorem hold then the integral in part (b) converges uniformly on compact subsets. Moreover, Π is an outer function in the upper half-plane and

$$|\Pi(x)|^2 = |2\pi\sigma'(x)|^{-1} \tag{1.7}$$

almost everywhere on $\mathbb{R}$ (see Lemma 8.6 in [10]), i.e.,

$$\Pi(\lambda) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{s-\lambda} - \frac{s}{s^2+1}\right)\log\sigma'(s)\,ds\right], \quad \lambda \in \mathbb{C}_+. \tag{1.8}$$

In addition, the function $[(\lambda+i)\Pi]^{-1}$ belongs to the Hardy space $H^2(\mathbb{C}_+)$.

1.4. Mate–Nevai–Totik Theorem

A. Máté, P. Nevai and V. Totik in 1991 proved [18] asymptotics of so-called Christoffel functions. Their result in the language of orthogonal polynomials can be considered as a refinement of part (c) of the Szegő theorem for points on the unit circle $\mathbb{T}$. Let us formulate this result. Let μ be a probability Borel measure on $[-\pi,\pi]$. For $z \in \mathbb{C}$, define the n-th Christoffel function w_n as follows:

$$w_n(\mu,z) = \min\left\{\frac{1}{2\pi|P(z)|^2}\int_{-\pi}^{\pi}|P(e^{it})|^2\,d\mu(t) \;\middle|\; P\text{-polynom.},\ \deg P < n\right\}.$$

Theorem C. (A. Máté, P. Nevai, V. Totik, [18]) *Let μ be a measure in the Szegő class on the unit circle. Then we have*

$$\lim_{n\to\infty} n w_n(\mu, e^{it}) = \mu'(t)$$

for almost every $t \in [-\pi,\pi]$.

The function w_n admits the representation

$$2\pi w_n(\mu,z) = \left(\sum_{k=0}^{n-1}|\varphi_k(z)|^2\right)^{-1}, \quad z \in \mathbb{C}, \tag{1.9}$$

where φ_n are the orthonormal polynomials associated with μ, see Theorem 11.3.1 in [25]. This allows us to formulate Theorem C in the following way.

Theorem D. *Let μ be a nontrivial probability Borel measure on $\mathbb{T}$ and φ_n be the orthogonal polynomials associated with μ. If μ belongs to the Szegő class, then for almost every $t \in [-\pi,\pi]$*

$$\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k(e^{it})|^2 \to \frac{1}{2\pi\mu'(t)}, \quad n\to\infty.$$

The core result of the present paper is the following theorem. It can be seen as a refinement of part (c) of the Krein theorem for points on the real line.

Theorem 2. *Consider a Krein system with a coefficient in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that its spectral measure σ is in the Szegő class on the real line. Then for almost every $x \in \mathbb{R}$*

$$\frac{1}{r}\int_0^r |P(s,x)|^2 ds \to \frac{1}{2\pi\sigma'(x)}, \quad r \to \infty.$$

1.5. Teplyaev's Conjectures

Notice that the convergence in part (c) of the Krein theorem holds only for some sequence of real numbers r_n while there is no such constraint in the part (c) of the Szegő theorem. A. Teplyaev proved [26] that the convergence $P_*(r,\cdot) \to \Pi$ holds for $a \in L^2(\mathbb{R}_+)$, see Theorem 1 in [26] or Theorem 11.1 in [10]. Additionally, it was proved that this result is sharp in the sense that it cannot be extended to any L^p for $p > 2$, see Theorem 3 in [26].

Teplyaev conjectured two positive results concerning the convergence of P_* to Π for a Krein system with a real-valued coefficient, see Conjecture 6.5, Conjecture 6.6 in [26]. The first conjecture was proved by Denisov in [10], we state it below:

Theorem E. (Lemma 8.6, [10]) *Assume that the conditions of the Krein theorem hold, the function a is real-valued, and a sequence $t_n \to \infty$ is such that $P(t_n,\lambda_0) \to 0$ for some $\lambda_0 \in \mathbb{C}_+$. Then $P_*(t_n,\lambda) \to \Pi(\lambda)$ uniformly on compact subsets of $\mathbb{C}_+$.*

The second conjecture concerns a convergence of P_* in average (i.e., in the Cesàro sense). We prove it in the present paper.

Theorem 3. *Assume a is a real-valued function and the assertions of the Krein theorem hold. Then Π is the limit in average of $P_*(r,\cdot)$, that is,*

$$\Pi(\lambda) = \lim_{r\to\infty} \frac{1}{r}\int_0^r P_*(\rho,\lambda)\, d\rho, \quad \lambda \in \mathbb{C}_+,$$

and the convergence is uniform on compact subsets of $\mathbb{C}_+$.

Notice that the limit in average of $P_*(r,\lambda)$ does not exist in general. In other words, there exists a function $a \in L^1_{\mathrm{loc}}$ such that the convergence in Theorem 3 does not hold, see discussion in the end of Section 6 in [26].

1.6. Open Problems

It is unknown whether the convergence in Theorem D or in Theorem 2 can be improved. Let us pose two related conjectures.

Conjecture II. *Let μ be a measure in the Szegő class on the unit circle and φ_n be the orthogonal polynomials associated with μ. Then for almost every $t \in [-\pi,\pi]$ we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi_k^*(e^{it}) \to \frac{1}{\sqrt{2\pi}}\Pi(e^{it}), \quad n \to \infty.$$

This may be connected with works [1,11,12].

Conjecture III. *Consider Krein system* (1.4). *Let* P, P_* *be its solitions and let* σ *be its spectral measure. Assume that* σ *is in the Szegő class on the real line. Then there exists a function* $\xi\colon \mathbb{R}_+ \to \mathbb{T}$ *such that the convergence*

$$\frac{1}{r}\int_0^r P(s,x)\xi(s)\,ds \to \Pi(x), \quad r \to \infty$$

holds for almost every $x \in \mathbb{R}$. *Additionally,* $\xi \equiv 1$ *if the coefficient of the Krein system is real-valued.*

The latter conjecture may be related to the results of [2].

1.7. Structure of the Paper

In the next section we collect some facts on Krein systems and prove Theorem 2. Proof of Teplyaev's conjecture, Theorem 3 is given in Sect. 3. In Sect. 4 we use Theorem 2 to prove Theorem 1.

2. Asymptotic Behaviour of Krein System Solutions on the Real Line: Proof of Theorem 2

2.1. Preliminaries

Christoffel–Darboux formula is one of the most important identities in the theory of orthogonal polynomials on the unit circle, for its statement and proof see Theorem 2.2.7 in [24] or Theorem 11.4.2 in [25]. We will use its continuous analog.

Lemma. (Christoffel–Darboux formula) *Let* P *and* P_* *be solutions of the Krein system* (1.4). *Then the equalities*

$$P(r,\lambda)\overline{P(r,\mu)} - P_*(r,\lambda)\overline{P_*(r,\mu)} = i(\lambda-\overline{\mu})\int_0^r P(s,\lambda)\overline{P(s,\mu)}\,ds, \quad (2.1)$$

$$|P_*(r,\lambda)|^2 = |P(r,\lambda)|^2 + 2\,\mathrm{Im}(\lambda)\int_0^r |P(s,\lambda)|^2\,ds \quad (2.2)$$

hold for any two complex numbers λ, μ.

Proof. Equation (2.2) follows from (2.1) by substituting $\mu = \lambda$; (2.1) can be proved by taking derivatives in both sides of the equality. □

Corollary 1. *For any* $\lambda \in \mathbb{C}$ *we have:*

1. $|P_*(r,\lambda)| = |P(r,\lambda)|$ *if* $\lambda \in \mathbb{R}$,
2. $|P_*(r,\lambda)| > |P(r,\lambda)|$ *if* $\mathrm{Im}(\lambda) > 0$,
3. $|P_*(r,\lambda)| < |P(r,\lambda)|$ *if* $\mathrm{Im}(\lambda) < 0$.

Proof. All three statements easily follow from (2.2). □

Corollary 2. $P_*(r,\cdot)$ *has no zeros in the half-plane* $\overline{\mathbb{C}_+} = \{z\colon \mathrm{Im}\, z \geqslant 0\}$.

Proof. From Part 2 of Corollary 1 it follows that there are no zeros in the open half-plane $\mathbb{C}_+$. Hence, to prove Corollary 2 we need to show the absence

of zeros on the real line. Assume the converse. Then there exists $x \in \mathbb{R}$ such that $P_*(r,x) = 0$. Part 1 of Corollary 1 gives

$$P_*(r,x) = P(r,x) = 0. \tag{2.3}$$

On the other hand, the functions $P_*(\cdot,x), P(\cdot,x)$ are the solutions of linear differential system (1.4) with non-zero initial values. Hence, (2.3) cannot hold. This contradiction concludes the proof. □

We will use the following statement (Lemma 8.5, [10]).

Lemma A. (Lemma 8.5, [10]) *If a measure σ belongs to the Szegő class and a sequence $r_n \to \infty$ is such that $P(r_n, \lambda_0) \to 0$ for some $\lambda_0 \in \mathbb{C}_+$, then the convergence $|P_*(r_n,\lambda)| \to |\Pi(\lambda)|$ holds uniformly in $\mathbb{C}_+$.*

Corollary 3. *From equation (2.2) and Lemma A we have*

$$|\Pi(\lambda)|^2 = 2\operatorname{Im}(\lambda) \int_0^\infty |P(s,\lambda)|^2 \, ds, \quad \lambda \in \mathbb{C}_+.$$

Remark. A similar relation in the theory of the orthogonal polynomials is given in Chapter 2.4 in [24].

2.2. An Equivalent form of Theorem 2

In this section we formulate Theorem 2 in a form similar to the original Máté–Nevai–Totik Theorem C. Let PW_r denote the Paley-Wiener space of entire functions f that can be represented in the form

$$f(z) = \int_0^r \varphi(s) e^{izs} \, ds, \quad z \in \mathbb{C}, \quad \varphi \in L^2[0,r].$$

Let a function m_r be given by

$$m_r(\sigma, z) = \inf \left\{ \tfrac{1}{|f(z)|^2} \textstyle\int_{-\infty}^{\infty} |f(t)|^2 \, d\sigma(t) \,\middle|\, f \in PW_r, f(z) \neq 0 \right\}, \quad z \in \mathbb{C}. \tag{2.4}$$

Similarly to (1.9), the function m_r can be calculated using the continuous analogs of the orthogonal polynomials. This is shown in the following lemma.

Lemma B. (S. Denisov, Lemma 8.2, [10]) *For any $z_0 \in \mathbb{C}$ the following is true:*

$$m_r(\sigma, z_0) = K_r(z_0,z_0)^{-1}, \qquad \text{where } K_r(z',z) = \int_0^r \overline{P(s,z')} P(s,z) \, ds.$$

Infimum in (2.4) is achieved on

$$f_r = \frac{K_r(z_0, \cdot)}{K_r(z_0,z_0)},$$

in particular, function f_r itself belongs to PW_r.

Lemma B gives

$$m_r(\sigma, z) = K_r(z, z)^{-1} = \left(\int_0^r |P(s, z)|^2 \, ds \right)^{-1},$$
$$rm_r(\sigma, z) = \left(\frac{1}{r} \int_0^r |P(s, z)|^2 \, ds \right)^{-1}.$$

Substituting the latter equality into Theorem 2, we get its equivalent form.

Theorem 4. *(Equivalent form of Theorem 2) Consider a Krein system with a coefficient in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that its spectral measure σ is in the Szegő class on the real line. Then for almost all $x \in \mathbb{R}$*

$$\lim_{r \to \infty} rm_r(\sigma, x) = 2\pi\sigma'(x).$$

We prove Theorem 4 by establishing two inequalities

$$\limsup_{r \to \infty} rm_r(\sigma, x) \leqslant 2\pi\sigma'(x), \tag{2.5}$$
$$\liminf_{r \to \infty} rm_r(\sigma, x) \geqslant 2\pi\sigma'(x) \tag{2.6}$$

in Sects. 2.3, 2.5 respectively.

2.3. Upper Bound

In this section we will use the following theorem. We give proof of this fact in Sect. 6.1.

Theorem F. *Let Φ be an even function such that $|\Phi(x)(1 + x^2)|$ is bounded and let σ be a measure satisfying (1.5). Then the convergence*

$$(\Phi_r * \sigma)(x) \to \sigma'(x) \int_{\mathbb{R}} \Phi(t) dt, \quad r \to \infty,$$

where $\Phi_r(t) = r\Phi(rt)$, holds for almost every $x \in \mathbb{R}$.

Lemma 1. *Let σ be the spectral measure of Krein system (1.4). Then we have*

$$\limsup_{r \to \infty} rm_r(\sigma, x) \leqslant 2\pi\sigma'(x)$$

for almost every $x \in \mathbb{R}$.

Remark. In the lemma we do not require σ to belong to the Szegő class.

Proof. Define the Fourier transform as $\mathcal{F}(f)(x) = \int_{\mathbb{R}} e^{itx} f(t)\, dt$. The inverse Fourier transform will take form $\mathcal{F}^{-1}(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} f(t)\, dt$. Consider functions

$$R(x) = \frac{e^{ix} - 1}{ix} = \mathcal{F}\left(\mathbf{1}_{[0,1]}\right)(x),$$
$$\Phi(x) = |R(x)|^2,$$

where $\mathbf{1}_{[0,1]}$ is the characteristic function of $[0, 1]$. Notice that

$$\mathcal{F}^{-1}(\Phi) = \mathcal{F}^{-1}\left(|R|^2\right) = \mathcal{F}^{-1}(R) * \mathcal{F}^{-1}\left(\overline{R}\right)$$
$$= \mathbf{1}_{[0,1]} * \mathbf{1}_{[-1,0]} = \begin{cases} 0, & |x| > 1 \\ 1 - |x|, & |x| \leqslant 1 \end{cases}.$$

In other words, Φ is a continuous Fejer kernel. By Plancherel theorem, we have

$$\int_{\mathbb{R}} \Phi(t)\,dt = \|R\|_{L_2}^2 = 2\pi\|\mathbf{1}_{[0,1]}\|_{L_2}^2 = 2\pi.$$

The function $f_{r,z}(x) = R(r(x-z))$ is in PW_r, and we have $f_{r,z}(z) = 1$. Hence

$$\begin{aligned} rm_r(\sigma, z) &\leqslant r\int_{-\infty}^{\infty} |f_{r,z}(x)|^2\,d\sigma(x) \\ &= r\int_{-\infty}^{\infty} |R(r(x-z))|^2\,d\sigma(x) = (\Phi_r * \sigma)(z). \end{aligned}$$

It only remains to pass to the upper limit as $r \to \infty$ and apply Theorem F to conclude the proof. □

2.4. Decay of Functions from Paley–Wiener Space

In this section we prove a result concerning functions of Paley–Wiener space; it is independent of the theory of Krein systems but will be useful to us. Let E_n denote the elementary Weierstrass factor,

$$E_n(z) = \begin{cases} (1-z), & n = 0, \\ (1-z)\exp\left(\frac{z}{1} + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right), & n > 0. \end{cases}$$

Theorem. (Hadamard factorization theorem, Section 4.2, [17]) *Let f be an entire function of finite order $\rho \geqslant 0$ and $\{a_n\}_{n=0}^{\infty}$ be its non-zero roots with multiplicity. Then f admits a factorization*

$$f(z) = e^{g(z)} z^m \prod_{k=0}^{\infty} E_d\left(\frac{z}{a_k}\right),$$

where g is a polynomial with degree not greater than ρ and $d = [\rho]$.

Lemma 2. *Let $f \in PW_r$. Suppose that f has only real zeros. Then we have*

$$|f(t)| \leqslant |f(t+\gamma i)|e^{\gamma r}$$

for every $t \in \mathbb{R}$ and $\gamma > 0$.

Proof. Denote by $\{a_n\}_{n=0}^{\infty}$ the zeros of f counting multiplicity. Functions from Paley-Wiener space are of order 1. Hence, by the Hadamard factorization theorem, f can be represented as

$$f(z) = e^{c_1 z + c_2} z^m \prod_{k=0}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k}}. \tag{2.7}$$

If $f(t) = 0$, the inequality in lemma is trivial. Otherwise we can write

$$\begin{aligned} \left|\frac{f(t+\gamma i)}{f(t)}\right| &= \left|e^{c_1\cdot\gamma i}\right| \left|\tfrac{t+\gamma i}{t}\right|^m \prod_{k=0}^{\infty} \left|\frac{1-\frac{t+\gamma i}{a_k}}{1-\frac{t}{a_k}} e^{\frac{\gamma i}{a_k}}\right| \\ &= \left|e^{c_1\cdot\gamma i}\right| \left|\tfrac{t+\gamma i}{t}\right|^m \prod_{k=0}^{\infty} \left|\frac{a_k - t - \gamma i}{a_k - t}\right| \\ &= \left|e^{c_1\cdot\gamma i}\right| \left|\tfrac{t+\gamma i}{t}\right|^m \prod_{k=0}^{\infty} \left|1 - \frac{\gamma i}{a_k - t}\right| \end{aligned}$$

$$= e^{-\operatorname{Im}(c_1)\gamma} \left|1 + \tfrac{\gamma^2}{t^2}\right|^{m/2} \prod_{k=0}^{\infty} \left|1 + \tfrac{\gamma^2}{(a_k - t)^2}\right|^{1/2} \geqslant e^{-\operatorname{Im}(c_1)\gamma}. \tag{2.8}$$

Choose a large real number N and substitute $-iN$ for z into (2.7).

$$|f(-iN)| = \left| e^{-c_1 iN + c_2} (-iN)^m \prod_{k=0}^{\infty} \left(1 - \frac{-iN}{a_k}\right) e^{\frac{-iN}{a_k}} \right|$$

$$= e^{N \operatorname{Im} c_1 + \operatorname{Re} c_2} N^m \prod_{k=0}^{\infty} \sqrt{1 + \frac{N^2}{a_k^2}} \geqslant e^{N \operatorname{Im} c_1 + \operatorname{Re} c_2}.$$

Since f belongs to PW_r, its exponential type is not greater than r. So

$$r \geqslant \limsup_{N \to \infty} \frac{\log |f(-iN)|}{N} \geqslant \limsup_{N \to \infty} \frac{N \operatorname{Im} c_1 + \operatorname{Re} c_2}{N} = \operatorname{Im} c_1.$$

The latter inequality together with (2.8) completes the proof. □

2.5. Lower Bound

Throughout this section we suppose that the measure σ belongs to the Szegő class on the real line. Consider the Szegő function D on $\mathbb{C}_+$ defined by

$$D = \left(\sqrt{2\pi}\Pi\right)^{-1}. \tag{2.9}$$

Now from (1.7) we get

$$|D(x)|^2 = \sigma'(x) \tag{2.10}$$

for almost all $x \in \mathbb{R}$. Taking this into account, we see that inequality (2.6) is equivalent to

$$\liminf_{r \to \infty} r m_r(\sigma, x) \geqslant 2\pi |D(x)|^2. \tag{2.11}$$

Next, let us introduce some constraints for the point $x = x_0$. First, assume that x_0 is a Lebesgue point of D, i.e.,

$$\frac{1}{h} \int\limits_{|x - x_0| < h} |D(x) - D(x_0)|\, dx \to 0, \quad h \to 0. \tag{2.12}$$

Second, assume that D has a non-tangential boundary value in x_0. Both of these constraints are satisfied on a set of a full Lebesgue measure; without loss of generality, we may assume that x_0 satisfies the constraints and $x_0 = 0$. Let us remember that f_r is a function for which a value of $m_r(\sigma, 0)$ is achieved (see Lemma B). That means

$$\int\limits_{\mathbb{R}} |f_r(x) D(x)|^2\, dx \leqslant \int\limits_{\mathbb{R}} |f_r(x)|^2\, d\sigma(x) = m_r(0) |f_r(0)|^2. \tag{2.13}$$

Fix $\varepsilon > 0$. To prove (2.11), we need to show that for a large enough r

$$r m_r(\sigma, 0) > (2\pi - 100\varepsilon) |D(0)|^2. \tag{2.14}$$

While proving (2.14) we may assume that

$$r m_r(\sigma, 0) < 2\pi |D(0)|^2, \tag{2.15}$$

otherwise there is nothing left to prove.

2.5.1. Auxiliary Lemmas. The first lemma can be proved using a standard argument involving Cauchy formula (see Sect. 6.2).

Lemma 3. *For every $z \in \mathbb{C}_+$ the following holds:*

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x)D(x)}{x-z}\,dx &= f_r(z)D(z),\\ \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x)D(x)}{x+z}\,dx &= 0. \end{aligned} \tag{2.16}$$

Lemma 4. *Let f_r be a function defined in Lemma B. For almost every $t \in \mathbb{R}$ and every $\gamma > 0$*

$$|f_r(t)| \leqslant |f_r(t+\gamma i)|e^{\gamma r}.$$

Proof. It is enough to show that f_r satisfies assertions of Lemma 2. Recall that by definition

$$f_r(\lambda) = \frac{K_r(0,\lambda)}{K_r(0,0)}, \qquad K_r(\lambda',\lambda) = \int_0^r \overline{P(s,\lambda')}P(s,\lambda)\,ds.$$

From Lemma B we already know that f_r belongs to the Paley-Wiener space PW_r. Hence, the only nontrivial question is the absence of zeros outside of the real line. Assume the converse. Then there exists $\lambda \notin \mathbb{R}$ such that $f_r(\lambda) = 0$. Therefore,

$$K_r(0,\lambda) = \int_0^r \overline{P(s,0)}P(s,\lambda)\,ds = 0$$

and the Christoffel–Darboux formula (2.1) gives

$$\begin{aligned} P(r,\lambda)\overline{P(r,0)} - P_*(r,\lambda)\overline{P_*(r,0)} &= 0,\\ |P(r,\lambda)P(r,0)| &= |P_*(r,\lambda)P_*(r,0)|. \end{aligned}$$

The latter equality contradicts Corollary 1. □

Lemma 5. *For a fixed number b the inequality*

$$|f_r(t)| < 25|f_r(0)|$$

holds for a large enough $r > 0$ and for all $t \in [-\frac{b}{r}, \frac{b}{r}]$.

Proof. The point $t + i/r$ lies in a cone $K = \{z\colon b|\operatorname{Im}(z)| \geqslant \operatorname{Re}(z)\}$. By assumption, D has a non-tangential boundary value at the point 0 and $D(0) \neq 0$. Hence, D is continuous at the point 0 inside the cone K and for a large enough r we have

$$\left|D\left(t+\frac{i}{r}\right)\right| > \frac{|D(0)|}{2}. \tag{2.17}$$

Lemma 3 gives

$$\left|f_r\left(t+\frac{i}{r}\right)D\left(t+\frac{i}{r}\right)\right|^2 = \left|\int_{\mathbb{R}} \frac{f_r(x)D(x)}{x-\left(t+\frac{i}{r}\right)}\,dx\right|^2$$

$$\leqslant \int_{\mathbb{R}} |f_r(x)D(x)|^2\, dx \int_{\mathbb{R}} \frac{1}{x^2+\frac{1}{r^2}}\, dx$$
$$\overset{(2.13)}{\leqslant} m_r(0)|f_r(0)|^2 \cdot \pi r \overset{(2.15)}{<} 2\pi^2 |D(0)f_r(0)|^2.$$

Hence,

$$|f_r(0)| > \frac{1}{\sqrt{2}\pi} \frac{\left|D\left(t+\frac{i}{r}\right)\right|}{|D(0)|} \left|f_r\left(t+\frac{i}{r}\right)\right|.$$

Applying Lemma 4 with $\gamma = \frac{1}{r}$ to bound $\left|f_r\left(t+\frac{i}{r}\right)\right|$, and inequality (2.17), we get

$$|f_r(0)| > \frac{1}{\sqrt{2}\pi} \frac{1}{2} e^{-1} |f_r(t)| > \frac{1}{25} |f_r(t)|.$$

□

2.5.2. Proof of (2.14). Recall that our aim is the proof of (2.14). Let H_r be a function defined by

$$H_r(x) = \frac{e^{-irx}-1}{x}.$$

Direct calculation (see Sect. 6.3) shows that for every $\delta > 0$ we have

$$\int_{-\infty}^{\infty} |H_r(x-\delta i)|^2\, dx < 2\pi r. \tag{2.18}$$

Put $\delta = \frac{\varepsilon}{r}$. Then the following inequalities hold:

$$\left|\int_{-\infty}^{\infty} f_r(x)H_r(x-\delta i)D(x)\, dx\right|^2 \leqslant \int_{-\infty}^{\infty} |f_r(x)D(x)|^2\, dx \int_{-\infty}^{\infty} |H_r(x-\delta i)|^2\, dx$$
$$\overset{(2.13)}{\leqslant} m_r(0)|f_r(0)|^2 \int_{-\infty}^{\infty} |H_r(x-\delta i)|^2\, dx$$
$$\leqslant m_r(0)|f_r(0)|^2 2\pi r.$$

Hence, we have

$$\left|\int_{-\infty}^{\infty} f_r(x)H_r(x-\delta i)D(x)\, dx\right| \leqslant |f_r(0)|\sqrt{2\pi r m_r(0)}. \tag{2.19}$$

Define I, I_1, I_2 by

$$I = \int_{-\infty}^{\infty} f_r(x)H_r(x-\delta i)D(x)\, dx$$
$$= \int_{-\infty}^{\infty} \frac{e^{-ir(x-\delta i)} f_r(x)D(x)}{x-\delta i}\, dx - \int_{-\infty}^{\infty} \frac{f_r(x)D(x)}{x-\delta i}\, dx = e^{-r\delta} I_2 - I_1.$$

Lemma 3 gives

$$I_1 = \int\limits_{-\infty}^{\infty} \frac{f_r(x)D(x)}{x-\delta i}dx = 2\pi i f_r(\delta i)D(\delta i). \tag{2.20}$$

Our further goal is to obtain the following estimate:

$$|I_2| \leqslant 4\varepsilon |f_r(0)D(0)|. \tag{2.21}$$

Before proving this, let us show how (2.21) can be used to establish inequality (2.14). Notice that

$$|I| = |e^{-r\delta}I_2 - I_1| \geqslant |I_1| - e^{-r\delta}|I_2| \geqslant |I_1| - |I_2|.$$

Substitution of (2.20), (2.19), (2.21) into the last inequality gives

$$|f_r(0)|\sqrt{2\pi r m_r(0)} \geqslant 2\pi |f_r(\delta i)D(\delta i)| - 4\varepsilon |f_r(0)D(0)|. \tag{2.22}$$

Since the function D has a non-tangential limit at the point 0, for a large enough r we have

$$|D(i\delta)| = \left|D\left(i\frac{\varepsilon}{r}\right)\right| > (1-\varepsilon)|D(0)|. \tag{2.23}$$

Moreover, from Lemma 4 we know

$$|f_r(\delta i)| \geqslant e^{-r\delta}|f_r(0)| = e^{-\varepsilon}|f_r(0)| \geqslant (1-\varepsilon)|f_r(0)|. \tag{2.24}$$

Finally, the substitution of (2.23) and (2.24) into inequality (2.22) finishes the proof:

$$\begin{aligned} |f_r(0)|\sqrt{2\pi r m_r(0)} &\geqslant 2\pi(1-\varepsilon)(1-\varepsilon)|f_r(0)D(0)| - 4\varepsilon|f_r(0)D(0)| \\ &\geqslant (2\pi - (4+4\pi)\varepsilon + 2\pi\varepsilon^2)|f_r(0)D(0)| \\ &> (2\pi - 20\varepsilon)|f_r(0)D(0)|, \\ r m_r(0) &\geqslant \frac{(2\pi-20\varepsilon)^2}{2\pi}|D(0)|^2 > (2\pi - 100\varepsilon)|D(0)|^2. \end{aligned}$$

Thus, it only remains to prove estimate (2.21). Consider one more integral $\tilde{I}_2$.

$$\tilde{I}_2 = \int\limits_{-\infty}^{\infty} \overline{\left(\frac{e^{-irx}f_r(x)}{x-\delta i}\right)}D(x)\,dx = \int\limits_{-\infty}^{\infty} \frac{e^{irx}\overline{f_r(x)}}{x+\delta i}D(x)\,dx.$$

The function f_r is in PW_r, so there exists $\varphi \in L^2[0,r]$ such that $f_r(x) = \int_0^r \varphi(t)e^{itx}dt$. Notice that

$$e^{irx}\overline{f_r(x)} = e^{irx}\int_0^r \overline{\varphi(t)}e^{-itx}dt = \int_0^r \overline{\varphi(t)}e^{i(r-t)x}dt = \int_0^r \overline{\varphi(r-t)}e^{itx}dt.$$

Hence, the function $e^{irx}\overline{f_r(x)}$ also belongs to PW_r and consequently to the Hardy space $H^2(\mathbb{C}_+)$. As we have mentioned before, the function $\frac{D}{x+\delta i}$ also belongs to $H^2(\mathbb{C}_+)$. Therefore

$$\frac{e^{irx}\overline{f_r}D}{x+\delta i} \in H^1(\mathbb{C}_+).$$

An integral of an $H^1(\mathbb{C}_+)$ function over the real line equals 0 (see Lemma 3.7 in [14]), therefore, $\tilde{I}_2 = 0$. By definition, put

$$K_1 = \left\{ x \in \mathbb{R} \colon |x| > \frac{b}{r} \right\}, \quad K_2 = \mathbb{R} \backslash K_1.$$

Split I_2 into two terms and consider them separately. We have

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} \frac{e^{-irx} f_r(x)}{x - \delta i} D(x)\, dx \\ &= \int_{K_1} \frac{e^{-irx} f_r(x)}{x - \delta i} D(x)\, dx + \int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i} D(x)\, dx = I_{21} + I_{22}. \end{aligned}$$

Similarly, $\tilde{I}_2$ is represented as a sum of $\tilde{I}_{21}$ and $\tilde{I}_{22}$. The following identity holds:

$$\begin{aligned} I_2/D(0) &= I_2/D(0) - \overline{\tilde{I}_2/D(0)} \\ &= I_{21}/D(0) + \left(I_{22}/D(0) - \textstyle\int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i}\, dx \right) \\ &\quad - \overline{\tilde{I}_{21}/D(0)} - \overline{\left(\tilde{I}_{22}/D(0) - \textstyle\int_{K_2} \overline{\left(\frac{e^{-irx} f_r(x)}{x - \delta i} \right)}\, dx \right)}. \end{aligned} \tag{2.25}$$

Let us bound each term in the right side.

$$|I_{21}|^2 = \left| \int_{K_1} \frac{e^{-irx} f_r(x) D(x)}{x - \delta i}\, dx \right|^2 \leqslant \int_{K_1} |f_r(x) D(x)|^2\, dx \int_{K_1} \left| \frac{e^{-irx}}{x - \delta i} \right|^2 dx,$$

$$\int_{K_1} |f_r(x) D(x)|^2\, dx \leqslant \int_{\mathbb{R}} |f_r(x) D(x)|^2\, dx \overset{s(2.13)}{\leqslant} m_r(0) |f_r(0)|^2,$$

$$\int_{K_1} \left| \frac{e^{-irx}}{x - \delta i} \right|^2 dx = \int\limits_{|x| > \frac{b}{r}} \frac{1}{x^2 + \delta^2}\, dx \leqslant 2 \int_{\frac{b}{r}}^{\infty} \frac{dx}{x^2} = 2r/b.$$

Thus,

$$|I_{21}|^2 \leqslant m_r(0) |f_r(0)|^2 \frac{2r}{b} \leqslant \frac{2}{b} |f_r(0)|^2 \left(r m_r(0) \right) \overset{(2.15)}{\leqslant} \frac{4\pi}{b} |f_r(0) D(0)|^2,$$

$$\frac{|I_{21}|}{|D(0)|} \leqslant \sqrt{\frac{4\pi}{b}} |f_r(0)|.$$

Choose b so large that $\frac{4\pi}{b} \leqslant \varepsilon^2$. Taking this into account, we obtain

$$\frac{|I_{21}|}{|D(0)|} \leqslant \varepsilon |f_r(0)|. \tag{2.26}$$

Next, we bound the second term in (2.25).

$$\begin{aligned} &\left| \frac{I_{22}}{D(0)} - \int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i}\, dx \right| \\ &= \frac{1}{|D(0)|} \left| \int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i} D(x)\, dx - \int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i} D(0)\, dx \right| \\ &\leqslant \frac{1}{|D(0)|} \sup_{K_2} \left| \frac{f_r(x)}{x - \delta i} \right| \int_{K_2} |D(x) - D(0)|\, dx. \end{aligned}$$

From Lemma 5 it follows that $|f_r(x)| < 25|f_r(0)|$ for every $x \in K_2$. Since 0 is a Lebesgue point of D (inequality (2.12) holds), we can choose r so large that

$$\int_{K_2} |D(x) - D(0)|\, dx = \int_{|x|<\frac{b}{r}} |D(x) - D(0)|\, dx < \frac{b}{r}\frac{\varepsilon^2}{25b}|D(0)|.$$

From the two latter inequalities we get

$$\left|\frac{I_{22}}{D(0)} - \int_{K_2} \frac{e^{-irx} f_r(x)}{x - \delta i}\, dx\right| \leqslant \tfrac{1}{|D(0)|} \cdot \tfrac{25|f_r(0)|}{\delta} \cdot \tfrac{b}{r}\tfrac{\varepsilon^2}{25b}|D(0)|$$
$$= \tfrac{\varepsilon^2}{\delta r}|f_r(0)| = \varepsilon|f_r(0)|. \tag{2.27}$$

The two remaining terms in (2.25) can be bounded in a similar way. Now substitution of (2.26), (2.27) and two similar inequalities for $\tilde{I}_2$ into (2.25) gives

$$I_2/D(0) \leqslant 4\varepsilon|f_r(0)|,$$

which is equivalent to (2.21).

3. Proof of Theorem 3

We begin by proving the average convergence for absolute value of P_*.

Lemma 6. *Assume that the coefficient a of Krein system (1.4) belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and the associated spectral measure σ belongs to the Szegő class on the real line. Then for every $\lambda \in \mathbb{C}_+$ we have*

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r |P_*(\rho, \lambda)|\, d\rho = |\Pi(\lambda)|\,.$$

Proof. The required equality is equivalent to two inequalities

$$\liminf_{r\to\infty} \frac{1}{r} \int_0^r |P_*(\rho, \lambda)|\, d\rho \geqslant |\Pi(\lambda)|\,, \tag{3.1}$$

$$\limsup_{r\to\infty} \frac{1}{r} \int_0^r |P_*(\rho, \lambda)|\, d\rho \leqslant |\Pi(\lambda)|\,. \tag{3.2}$$

Pass to the lower limit as $r \to \infty$ in Christoffel–Darboux formula (2.2):

$$\liminf_{r\to\infty} |P_*(r, \lambda)|^2 = \liminf_{r\to\infty} \left(|P(r, \lambda)|^2 + 2\,\mathrm{Im}(\lambda) \int_0^r |P(s, \lambda)|^2\, ds\right).$$

Part (b) of the Krein theorem states that $P(\cdot, \lambda)$ belongs to $L^2(\mathbb{R}_+)$. Hence,

$$\liminf_{r\to\infty} |P(r, \lambda)|^2 = 0.$$

The second term tends to the integral from 0 to ∞; by Corollary 3 this integral is equal to $|\Pi(\lambda)|^2$. Thus,

$$\liminf_{r\to\infty} |P_*(r, \lambda)| = |\Pi(\lambda)|.$$

Inequality (3.1) follows from the last equality by the direct integration. To prove (3.2), integrate and divide by r Christoffel–Darboux formula (2.2):

$$\begin{aligned}\frac{1}{r}\int_0^r |P_*(\rho,\lambda)|^2\,d\rho &= \frac{1}{r}\int_0^r \left(|P(\rho,\lambda)|^2 + 2\operatorname{Im}(\lambda)\int_0^\rho |P(s,\lambda)|^2\,ds\right)d\rho\\ &= \frac{1}{r}\int_0^r |P(\rho,\lambda)|^2\,d\rho + 2\operatorname{Im}(\lambda)\frac{1}{r}\int_0^r\int_0^\rho |P(s,\lambda)|^2\,ds\,d\rho\\ &\leqslant \frac{1}{r}\int_0^\infty |P(\rho,\lambda)|^2\,d\rho + 2\operatorname{Im}(\lambda)\frac{1}{r}\int_0^r\int_0^\infty |P(s,\lambda)|^2\,ds\,d\rho\\ &= \left(\frac{1}{2r\operatorname{Im}(\lambda)}+1\right)2\operatorname{Im}(\lambda)\int_0^\infty |P(s,\lambda)|^2\,ds\\ &\overset{\text{Cor.3}}{=} \left(\frac{1}{2r\operatorname{Im}(\lambda)}+1\right)|\Pi(\lambda)|^2.\end{aligned}$$

Now, the Cauchy-Schwarz inequality gives

$$\frac{1}{r}\int_0^r |P_*(\rho,\lambda)|\,d\rho \leqslant \sqrt{\frac{1}{r}\int_0^r |P_*(\rho,\lambda)|^2\,d\rho} \leqslant \sqrt{1+\frac{1}{2r\operatorname{Im}(\lambda)}}\,|\Pi(\lambda)|. \tag{3.3}$$

Passing to the upper limit as $r\to\infty$ concludes the proof. □

Corollary 4. *Suppose that the spectral measure σ belongs to the Szegő class and the function a is real-valued. Then*

$$\lim_{r\to\infty}\frac{1}{r}\int_0^r P_*(\rho,i)\,d\rho = \Pi(i).$$

Proof. Substituting $\lambda = i$ in (1.8), we obtain

$$\begin{aligned}\Pi(i) &= \frac{1}{\sqrt{2\pi}}\exp\left[\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{(1+si)\ln\sigma'(s)}{(i-s)(1+s^2)}\,ds\right]\\ &= \frac{1}{\sqrt{2\pi}}\exp\left[\frac{-1}{2\pi}\int_{-\infty}^\infty \frac{\ln\sigma'(s)}{(1+s^2)}\,ds\right].\end{aligned}$$

Hence, $\Pi(i)$ is a positive real number. System (1.4) with $\lambda = i$ has real coefficients, real initial condition and consequently the solution $P_*(r,i)$ is real for every r. By Corollary 2, $P_*(r,i)$ is not equal to zero for $r > 0$; $P_*(0,i) > 0$. Therefore, $P_*(r,i)$ must be positive for every $r \geqslant 0$ and

$$\lim_{r\to\infty}\frac{1}{r}\int_0^r P_*(\rho,i)\,d\rho = \lim_{r\to\infty}\frac{1}{r}\int_0^r |P_*(\rho,i)|\,d\rho = |\Pi(i)| = \Pi(i).$$

□

Now we can proceed directly to the proof of Theorem 3.

Proof of Theorem 3. Consider a family $\mathcal{S}$ of the analytic functions

$$F_r(\lambda) = \frac{1}{r}\int_0^r P_*(\rho,\lambda)\,d\rho, \quad r > 1.$$

Notice that

$$|F_r(\lambda)| = \left|\frac{1}{r}\int_0^r P_*(\rho,\lambda)\,d\rho\right| \leqslant \frac{1}{r}\int_0^r |P_*(\rho,\lambda)|\,d\rho \overset{(3.3)}{\leqslant} \sqrt{1+\frac{1}{2r\,\mathrm{Im}(\lambda)}}|\Pi(\lambda)|. \tag{3.4}$$

Consider an arbitrary compact set $D \subset \mathbb{C}_+$. Since $\mathrm{Im}\,\lambda$ is separated from 0 in D, we have

$$\sup_{\lambda\in D,r}|F_r(\lambda)| \leqslant \sqrt{1+\frac{1}{2\,\mathrm{dist}(D,\mathbb{R})}}\cdot\sup_{\lambda\in D}|\Pi(\lambda)| < \infty.$$

Hence, the family $\mathcal{S}$ is uniformly bounded on compact subsets and, by Montel theorem, every infinite subset of $\mathcal{S}$ has a convergent subsequence. Let a sequence $r_n\to\infty$ be such that a sequence F_{r_n} is convergent and $\lim F_{r_n} = \tilde{F}$ as $n\to\infty$. Inequality (3.4) gives

$$|\tilde{F}(\lambda)| = \lim_{n\to\infty}|F_{r_n}(\lambda)| \leqslant |\Pi(\lambda)|$$

for every point $\lambda\in\mathbb{C}_+$. On the other hand, by Corollary 4 we have

$$\tilde{F}(i) = \lim_{n\to\infty}F_{r_n}(i) = \lim_{n\to\infty}\frac{1}{r_n}\int_0^{r_n}P_*(\rho,i)\,d\rho = \Pi(i).$$

Applying the maximum modulus principle, we conclude that $\tilde{F}(\lambda)=\Pi(\lambda)$ for every $\lambda\in\mathbb{C}_+$. Any partial limit of F_r as $r\to\infty$ coincides with Π, so F_r has a limit equal to Π. This completes the proof. □

4. Dirac Equation: Proof of Theorem 1

Let us show how to derive Theorem 1 from Theorem 2 using an appropriate transformation of Dirac differential equation (1.1) to Krein system (1.4). To do that, we need some notation. Consider system (1.4) and let the functions φ,ψ,p,q be defined as follows:

$$\varphi(r,\lambda) = \tfrac{\exp(-i\lambda r)}{2}\left[P(2r,\lambda)+P_*(2r,\lambda)\right],\quad \varphi(0,\lambda)=1, \tag{4.1}$$

$$\psi(r,\lambda) = \tfrac{\exp(-i\lambda r)}{2i}\left[P(2r,\lambda)-P_*(2r,\lambda)\right],\quad \psi(0,\lambda)=0,$$

$$p(r) = -2\,\mathrm{Re}\,a(2r),\qquad q(r) = 2\,\mathrm{Im}\,a(2r). \tag{4.2}$$

Direct calculations give that functions φ and ψ are the solutions of the differential equation

$$\lambda\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = J\begin{pmatrix}\varphi\\ \psi\end{pmatrix}' + Q\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\ \varphi(0,\lambda)=1,\ \psi(0,\lambda)=0, \tag{4.3}$$

where

$$J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad Q=\begin{pmatrix}-q & p\\ p & q\end{pmatrix}.$$

Now we proceed to the proof of Theorem 1. Construct a Krein system with the coefficient $a(r) = -\frac{1}{2}p(r/2)+\frac{i}{2}q(r/2)$, where p,q are the potentials in equation (1.1). Uniqueness theorem and (4.3) imply that functions φ and ψ defined by (4.1) and (4.2) respectively are the unique solutions of equation

(1.1). Since potentials p, q are in $L^2(\mathbb{R}_+)$, coefficient a is also in $L^2(\mathbb{R}_+)$. Spectral measure of the Krein system with a square-summable coefficient belongs to the Szegő class (see Theorem 11.1 in [10]). Hence, Theorem 2 can be applied and the convergence

$$\frac{1}{r}\int_0^r |P(s,x)|^2\,ds \to (2\pi\sigma'(x))^{-1}, \qquad \text{as } r\to\infty,$$

holds for almost every $x\in\mathbb{R}$. Consequently, for almost every $x\in\mathbb{R}$ we have

$$\sup_{r>0}\frac{1}{r}\int_0^r |P(s,x)|^2\,ds<\infty.$$

Finally,

$$\begin{aligned}
\sup_{r>0}\frac{1}{r}\int_0^r |\varphi(s,x)|^2 ds &= \sup_{r>0}\frac{1}{r}\int_0^r \left|\frac{\exp(-ixs)}{2}\left[P(2s,x)+P_*(2s,x)\right]\right|^2 ds\\
&= \sup_{r>0}\frac{1}{r}\int_0^r \left|\frac{P(2s,x)+P_*(2s,x)}{2}\right|^2 ds\\
&\leqslant \sup_{r>0}\frac{1}{r}\int_0^r \frac{|P(2s,x)|^2+|P_*(2s,x)|^2}{2}\,ds\\
&= \sup_{r>0}\frac{1}{r}\int_0^r |P(2s,x)|^2\,ds<\infty.
\end{aligned}$$

The equality on the last line follows from Corollary 1 to Christoffel–Darboux formula (see Sect. 2.1). The same argument works for ψ.

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6. Appendix

6.1. Proof of Theorem F

Recall that Φ is an even function such that $|\Phi(x)(1+x^2)|$ is bounded and σ is a measure satisfying (1.5). We need to prove that

$$(\Phi_r * \sigma)(x) \to \sigma'(x)\int_{\mathbb{R}}\Phi(t)dt, \quad r\to\infty, \tag{6.1}$$

for almost every $x\in\mathbb{R}$, where $\Phi_r(t)=r\Phi(rt)$. By a restriction of σ to some interval one can assume that σ is finite. Now to prove (6.1) one can deal with the density and the singular part of σ individually: (6.1) for the density follows from the basic property of the approximate identity, see Theorem 8.15 in [13]; for the singular part one can adapt the proof of Lemma 5.4 in [14].

6.2. Proof of Lemma 3

We need to prove that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x)D(x)}{x-z}\,dx &= f_r(z)D(z),\\ \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x)D(x)}{x+z}\,dx &= 0.\end{aligned} \tag{6.2}$$

We will need the following

Lemma 7. *For every point $z \in \mathbb{C}_+$ the following inequality holds:*

$$|f_r(z)D(z)| \leqslant K_r(0,0)^{-\frac{1}{2}}(4\pi \operatorname{Im}(z))^{-\frac{1}{2}},$$

in particular, for any $c > 0$ the function $f_r D$ is bounded in the half-plane $\{z\colon \operatorname{Im}(z) > c\}$.

Proof.

$$\begin{aligned}|f_r(z)| &= \left|\frac{K_r(0,z)}{K_r(0,0)}\right| = \left|\frac{\int_0^r \overline{P(s,0)}P(s,z)\,ds}{\int_0^r |P(s,0)|^2\,ds}\right|\\ &\leqslant \frac{\left(\int_0^r |P(s,0)|^2\,ds\right)^{\frac{1}{2}}\left(\int_0^r |P(s,z)|^2\,ds\right)^{\frac{1}{2}}}{\int_0^r |P(s,0)|^2\,ds} \leqslant K_r(0,0)^{-\frac{1}{2}}\left(\frac{|\Pi(z)|^2}{2\operatorname{Im}(z)}\right)^{\frac{1}{2}}\\ &\overset{(2.9)}{=} K_r(0,0)^{-\frac{1}{2}}\left(\frac{1}{2\operatorname{Im}(z)\cdot 2\pi|D(z)|^2}\right)^{\frac{1}{2}}\\ &= K_r(0,0)^{-\frac{1}{2}}(4\pi\operatorname{Im}(z))^{-\frac{1}{2}}|D(z)|^{-1}.\end{aligned}$$

The second inequality on the second line follows from Corollary 3. Multiplying the last inequality by $|D(z)|$, we obtain the required. □

Remark. If the case when $f_r D \in H^p(\mathbb{C}_+)$ for some $p \geqslant 1$, the conclusion of Lemma 3 is known (see p. 116 in [15]). It is not our case but function $f_r D$ satisfies a "boundedness" condition from Lemma 7.

Proof of Lemma 3. We have already mentioned that the function

$$\frac{\sqrt{2\pi}^{-1}\Pi^{-1}}{\lambda+i} = \frac{D}{\lambda+i}$$

is in the Hardy space $H^2(\mathbb{C}_+)$. Hence, for a fixed $z \in \mathbb{C}_+$ the function $\frac{D}{\lambda+z}$ also belongs to $H^2(\mathbb{C}_+)$. The function f_r belongs to the Paley-Wiener space and consequently to $H^2(\mathbb{C}_+)$. Therefore both integrals in the lemma are absolutely convergent and define analytic functions in $\mathbb{C}_+$. Moreover, $f_r\frac{D}{\lambda+z}$ is a multiplication of two functions from $H^2(\mathbb{C}_+)$ and consequently it belongs to $H^1(\mathbb{C}_+)$. Integral of an $H^1(\mathbb{C}_+)$ function over the real line equals 0 (see p. 122 in [15] or Lemma 3.7 in [14]). Thus, the second equality in lemma is proved.

By Lemma 7, the function $G_\varepsilon(\lambda) = f_r(\lambda + \varepsilon i)D(\lambda + \varepsilon i)$ is bounded in the upper half-plane $\mathbb{C}_+$. Proceeding as above, we deduce that for all $z \in \mathbb{C}_+$

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{G_\varepsilon(x)}{x+z}\,dx = 0.$$

Next, the following chain of equalities holds:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x+\varepsilon i)D(x+\varepsilon i)}{x-z}\,dx &= \tfrac{1}{2\pi i}\textstyle\int_{\mathbb{R}} \frac{G_\varepsilon(x)}{x-z}\,dx \\ = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{G_\varepsilon(x)}{x-z}\,dx - \tfrac{1}{2\pi i}\textstyle\int_{\mathbb{R}} \frac{G_\varepsilon(x)}{x+z}\,dx &= \tfrac{1}{2\pi i}\textstyle\int_{\mathbb{R}} \frac{2zG_\varepsilon(x)}{x^2-z^2}\,dx.\end{aligned} \tag{6.3}$$

The latter integral can be calculated by the calculus of residues. Take a large positive number R and consider a contour C_R consisting of the segment $[-R, R]$ and the half-circle Γ_R connecting points R and $-R$ in the upper half-plane.

$$\begin{aligned}G_\varepsilon(z) &= \operatorname{Res}_z\left(\tfrac{2zG_\varepsilon(x)}{x^2-z^2}\right) = \tfrac{1}{2\pi i}\int\limits_{C_R} \tfrac{2zG_\varepsilon(x)}{x^2-z^2}dx \\ &= \tfrac{1}{2\pi i}\int\limits_{[-R,R]} \tfrac{2zG_\varepsilon(x)}{x^2-z^2}dx + \tfrac{1}{2\pi i}\int\limits_{\Gamma_R} \tfrac{2zG_\varepsilon(x)}{x^2-z^2}dx.\end{aligned} \tag{6.4}$$

The first term in the latter sum has a limit as $R \to \infty$.

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int\limits_{[-R,R]} \frac{2zG_\varepsilon(x)}{x^2-z^2}\,dx = \frac{1}{2\pi i}\int\limits_{\mathbb{R}} \frac{2zG_\varepsilon(x)}{x^2-z^2}\,dx.$$

Since the function G_ε is bounded, the second term tends to zero as $R \to \infty$:

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\Gamma_R} \frac{2zG_\varepsilon(x)}{x^2-z^2}\,dx\right| &\leqslant \frac{1}{2\pi}\int_0^\pi \left|\frac{2zG_\varepsilon(Re^{i\theta})}{R^2e^{2i\theta}-z^2}\right| R\,d\theta \\ &\leqslant \frac{R|z|}{R^2-|z|^2}\sup_\theta |G_\varepsilon(Re^{i\theta})| = O\left(\frac{1}{R}\right).\end{aligned}$$

We can therefore pass to the limit $R \to \infty$ in (6.4) and obtain

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{2zG_\varepsilon(x)}{x^2-z^2}\,dx = G_\varepsilon(z).$$

Substituting this into (6.3) we get

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f_r(x+\varepsilon i)D(x+\varepsilon i)}{x-z}\,dx = G_\varepsilon(z) = f_r(z+i\varepsilon)D(z+i\varepsilon).$$

To prove equality (6.2) it only remains to pass to the limit as ε tends to 0. The right part of the equality tends to $f_r(z)D(z)$ so we need to evaluate how the left side differs from the integral in the lemma.

$$\begin{aligned}&\left|\int_{\mathbb{R}} \frac{f_r(x+\varepsilon i)D(x+\varepsilon i)}{x-z}\,dx - \int_{\mathbb{R}} \frac{f_r(x)D(x)}{x-z}\,dx\right| \\ &= \left|\int_{\mathbb{R}} f_r(x+i\varepsilon)\frac{D(x+\varepsilon i)-D(x)}{x-z}\,dx + \int_{\mathbb{R}} D(x)\frac{f_r(x+i\varepsilon)-f_r(x)}{x-z}\,dx\right|\end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_{\mathbb{R}} \left| f_r(x+i\varepsilon) \frac{D(x+\varepsilon i)-D(x)}{x-z} \right| dx + \int_{\mathbb{R}} \left| D(x) \frac{f_r(x+i\varepsilon)-f_r(x)}{x-z} \right| dx \\
&\leqslant \|f_r(x+\varepsilon i)\|_2 \cdot \left\| \frac{x+i}{x-z} \right\|_\infty \left\| \frac{D(x+\varepsilon i)-D(x)}{x+i} \right\|_2 \\
&\quad + \|f_r(x+i\varepsilon)-f_r(x)\|_2 \cdot \left\| \frac{x+i}{x-z} \right\|_\infty \left\| \frac{D(x)}{x+i} \right\|_2 \\
&\leqslant C \left(\left\| \frac{D(x+\varepsilon i)-D(x)}{x+i} \right\|_2 + \|f_r(x+i\varepsilon)-f_r(x)\|_2 \right) \\
&\leqslant C \left(\left\| \frac{D(x+\varepsilon i)}{x+i} - \frac{D(x+\varepsilon i)}{x+\varepsilon i+i} \right\|_2 + \left\| \frac{D(x+\varepsilon i)}{x+\varepsilon i+i} - \frac{D(x)}{x+i} \right\|_2 \right. \\
&\quad \left. + \|f_r(x+i\varepsilon)-f_r(x)\|_2 \right).
\end{aligned}
$$

The first term can be bounded in the following way:

$$
\left\| \frac{D(x+\varepsilon i)}{x+i} - \frac{D(x+\varepsilon i)}{x+\varepsilon i+i} \right\|_2 = \varepsilon \left\| \frac{D(x+\varepsilon i)}{(x+\varepsilon i+i)(x+i)} \right\|_2 \leqslant \varepsilon \left\| \frac{D(x)}{x+i} \right\|_2 \to 0.
$$

From a general property of functions in $H^2(\mathbb{C}_+)$ (see Theorem 3.1 in [14] or Section 19.2 in [17]), it follows that the second and the third terms in the bracket tend to zero as $\varepsilon \to 0$. □

6.3. Proof of (2.18)

Proof. We need to prove that for every $\delta > 0$ the following holds:

$$
\int_{-\infty}^{\infty} |H_r(x-\delta i)|^2 \, dx < 2\pi r,
$$

where $H_r(x) = \frac{e^{-irx}-1}{x}$. Recall the formula of the Fourier transform of $\frac{1}{x^2+1}$:

$$
\mathcal{F}\left(\frac{1}{x^2+1} \right)(w) = \int_{-\infty}^{\infty} \frac{e^{iwx}}{x^2+1} \, dx = \pi e^{-|w|}.
$$

With that in mind, the integral can be easily estimated, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} |H_r(x-\delta i)|^2 \, dx &= \int_{-\infty}^{\infty} \frac{\left| e^{-r\delta} e^{irx} - 1 \right|^2}{x^2+\delta^2} \, dx \\
&= \int_{-\infty}^{\infty} \frac{e^{-2r\delta} + 1 - e^{-r\delta}(e^{irx} + e^{-irx})}{x^2+\delta^2} \, dx \\
&= \frac{1}{\delta} \left[(e^{-2r\delta}+1) \int_{-\infty}^{\infty} \frac{dy}{y^2+1} - e^{-r\delta} \int_{-\infty}^{\infty} \frac{e^{ir\delta y} + e^{-ir\delta y}}{y^2+1} \, dy \right] \\
&= \frac{1}{\delta} \left[(e^{-2r\delta}+1)\pi - 2e^{-r\delta}\pi e^{-r\delta} \right] = \pi \frac{1-e^{-2r\delta}}{\delta} < 2\pi r.
\end{aligned}
$$

□

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